# Session 4. Approximate solution of equations: isolation of roots and bisection method

# Introduction:

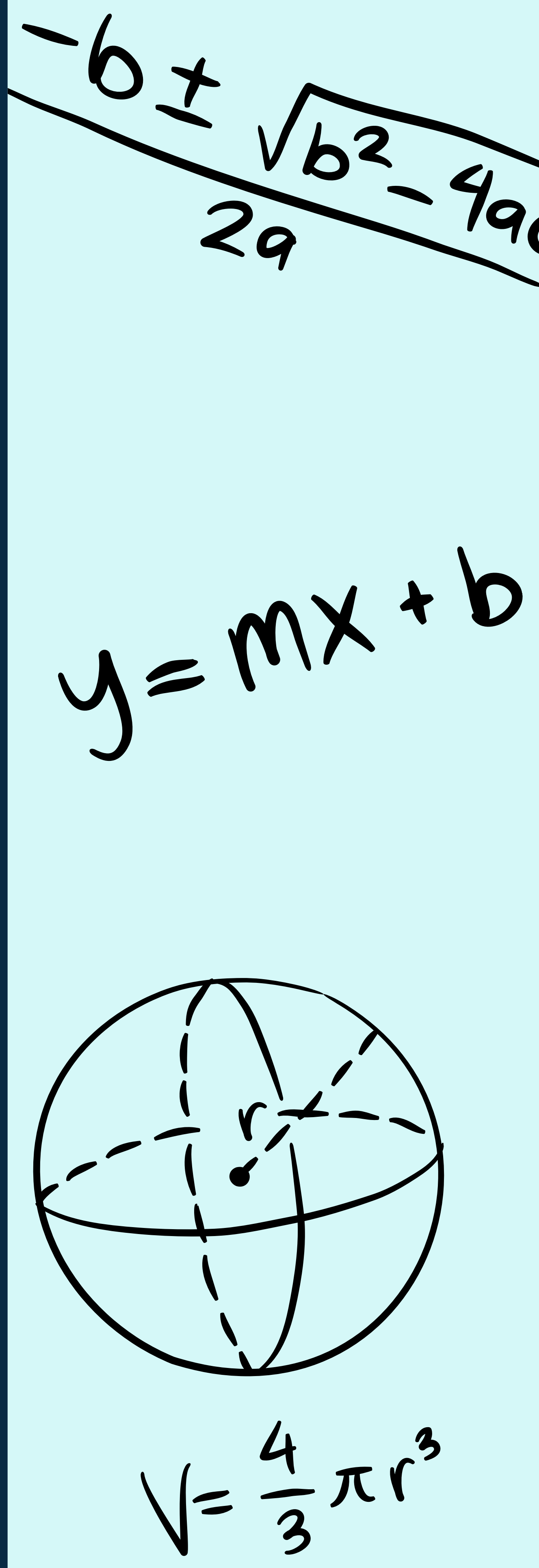
Given a continuous function f and the equation

$$f(x) = 0,$$

The **objective** is

- To determine how many real solutions the equation has;
- To find intervals [a, b] that contain only one solution.

With Matlab, we can make use of the graph of f to solve both problems.



# STEPS TO FOLLOW:

**plot f(x)**

**Choice of intervals**

**The command roots**

only for polynomials

# EXAMPLE:

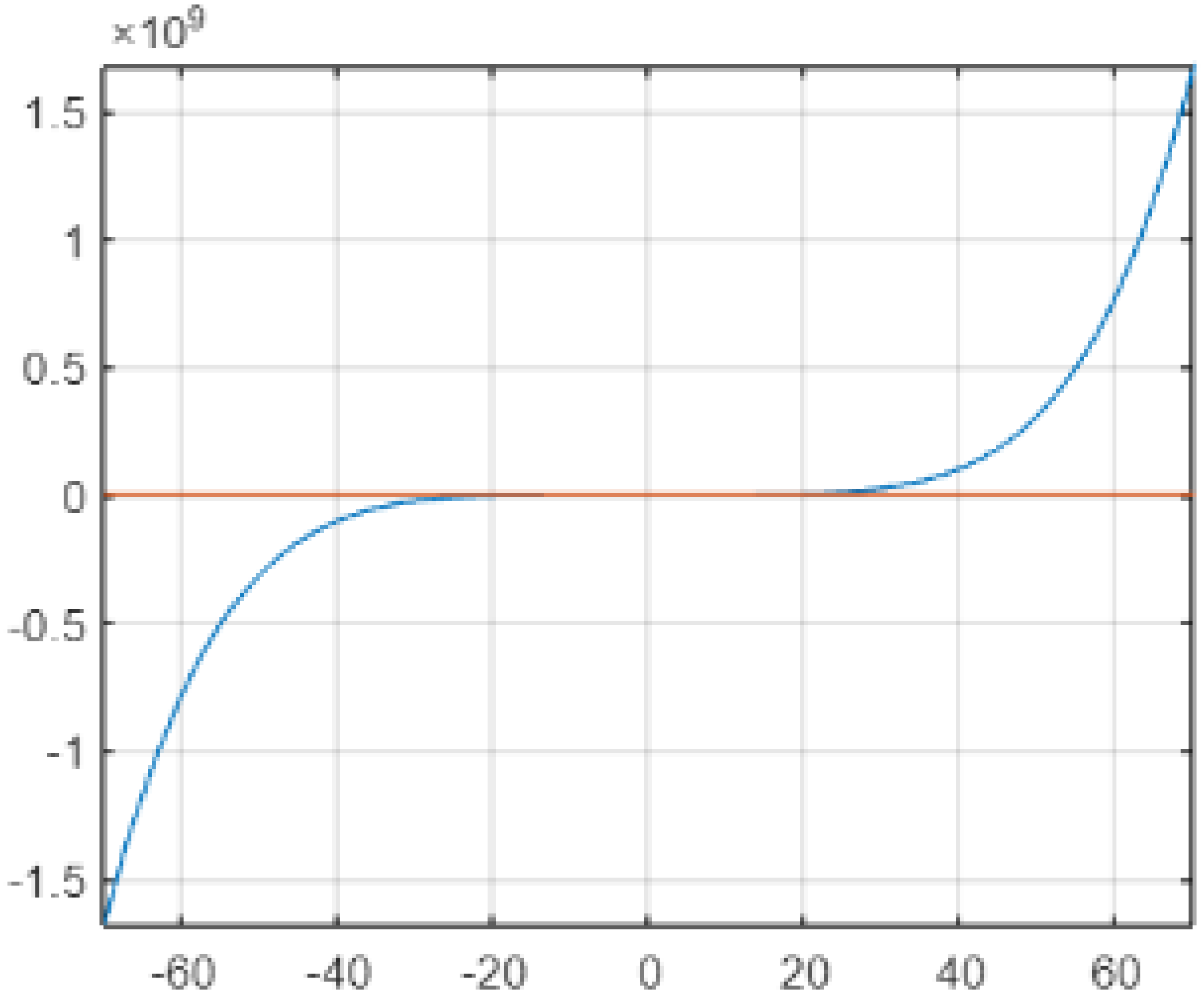
**Determine the number of real solutions of the following equation and find intervals of length 1 in which each of these solutions is unique**

$$f(x) = x^5 + 20x - 100$$

# plot f(x)

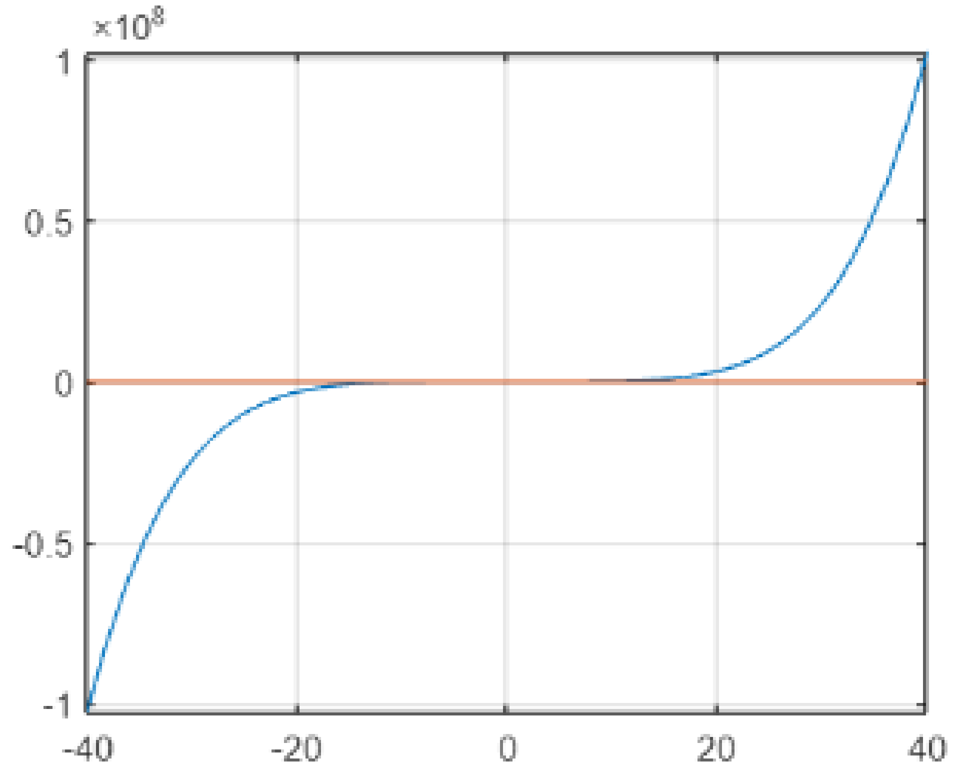
```
% Function b
syms f(x)
f(x)= x^5 + 20*x -100
fplot({f,0} ,[-70,70]), grid
```

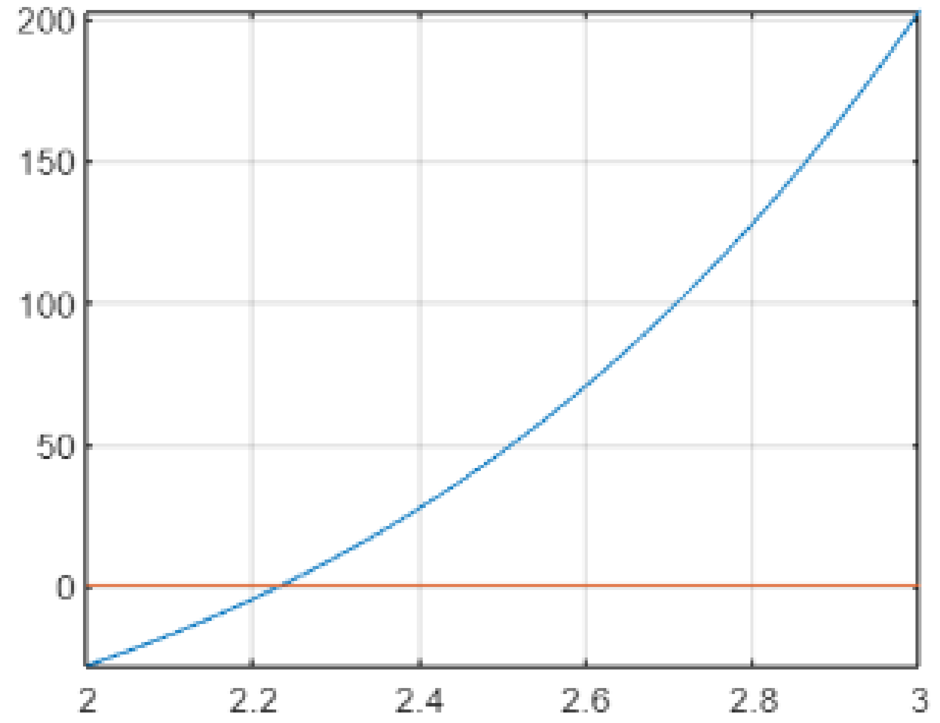
$$f(x) = x^5 + 20\,x - 100$$

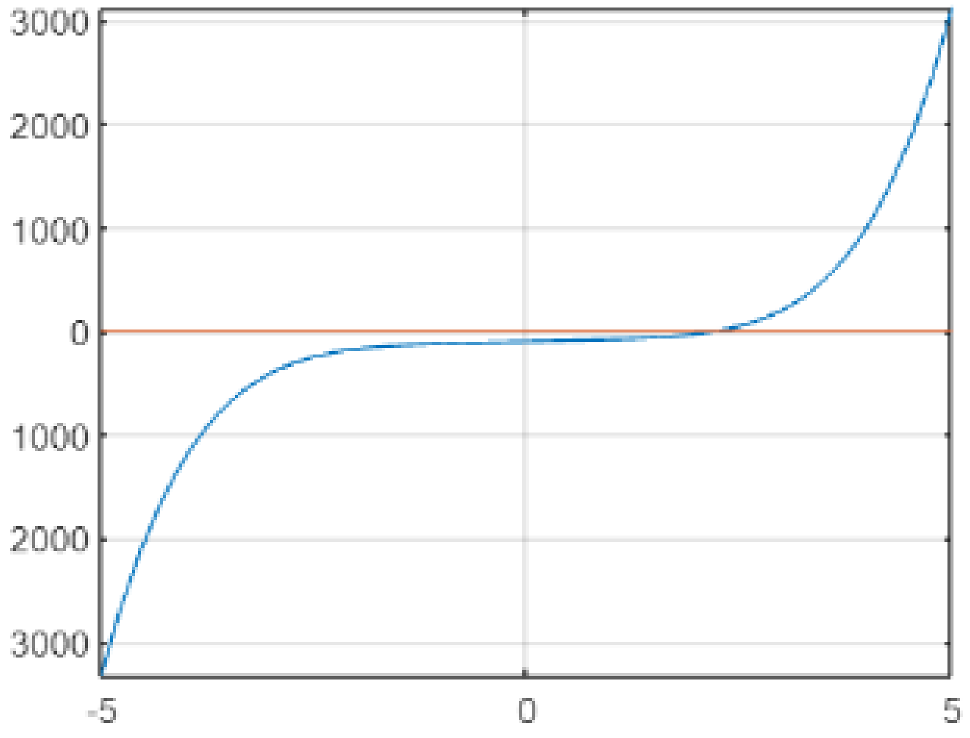


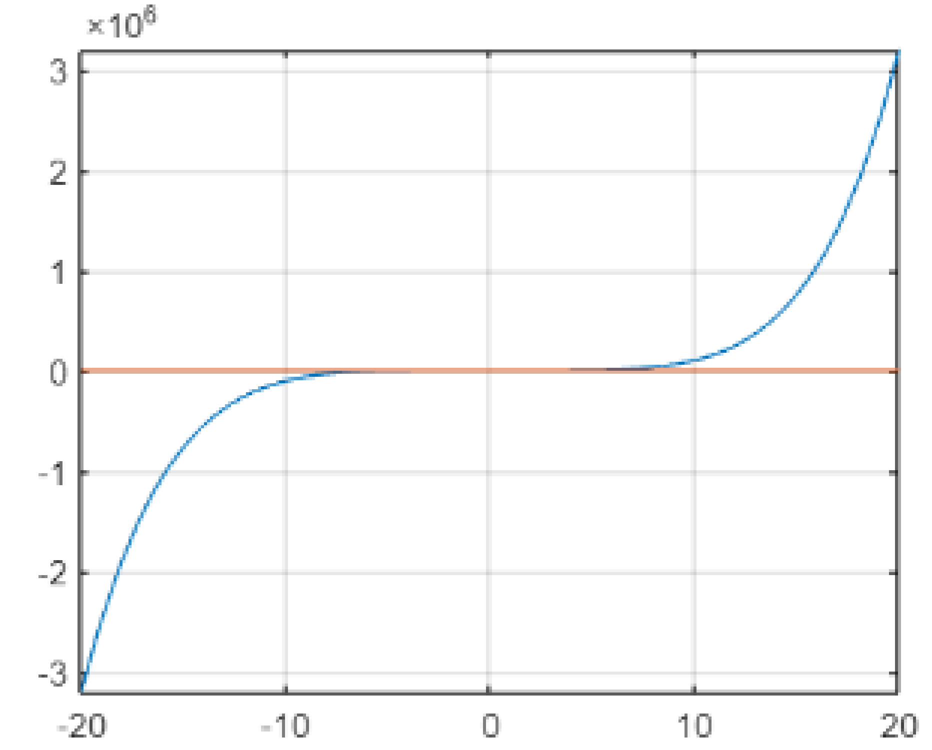
# Choice of intervals

```
fplot({f,0},[-40,40]), grid
fplot({f,0},[-20,20]), grid
fplot({f,0},[-5,5]), grid
fplot({f,0},[2,3]), grid
```



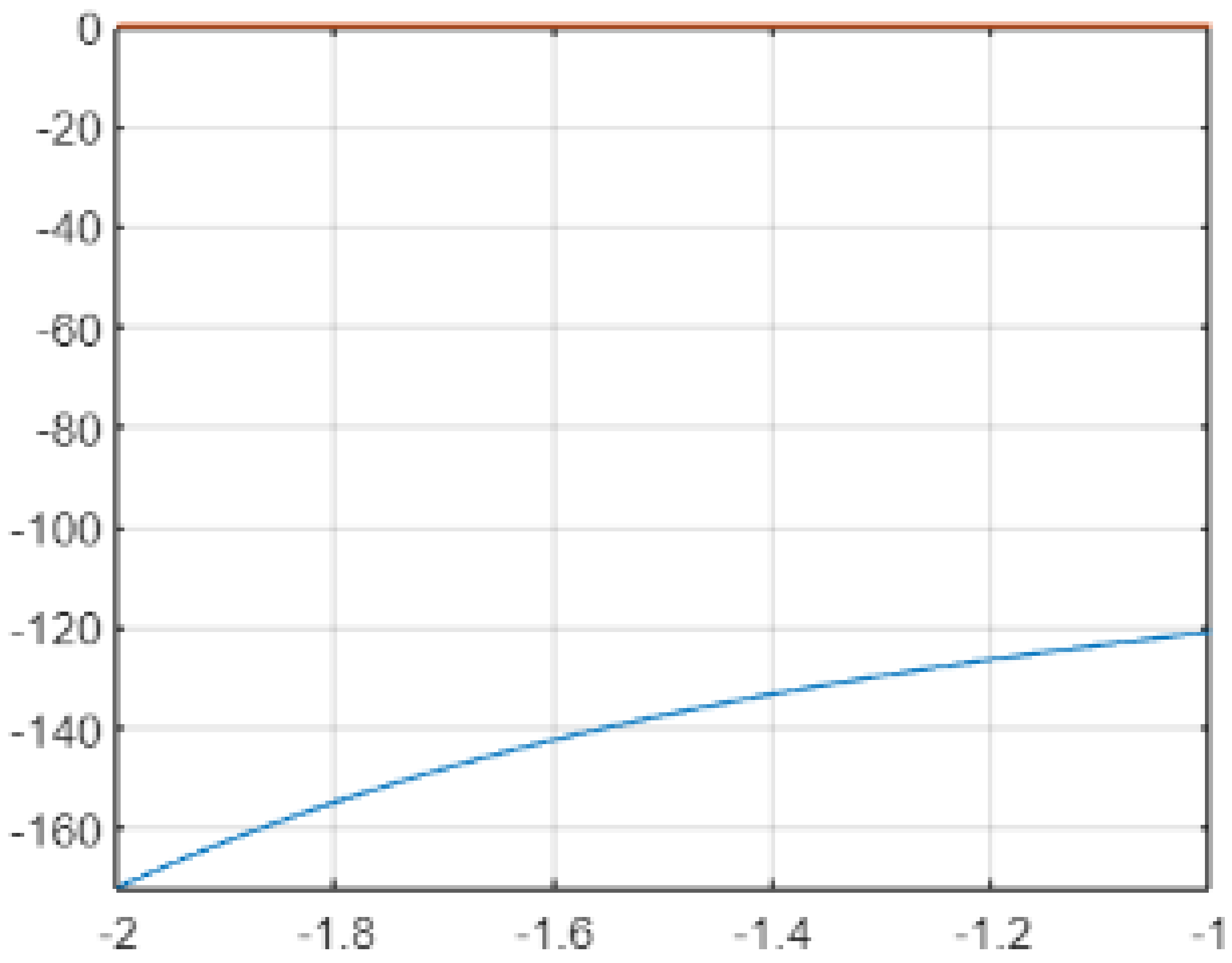



## Choice of intervals

**The only dubious zone**

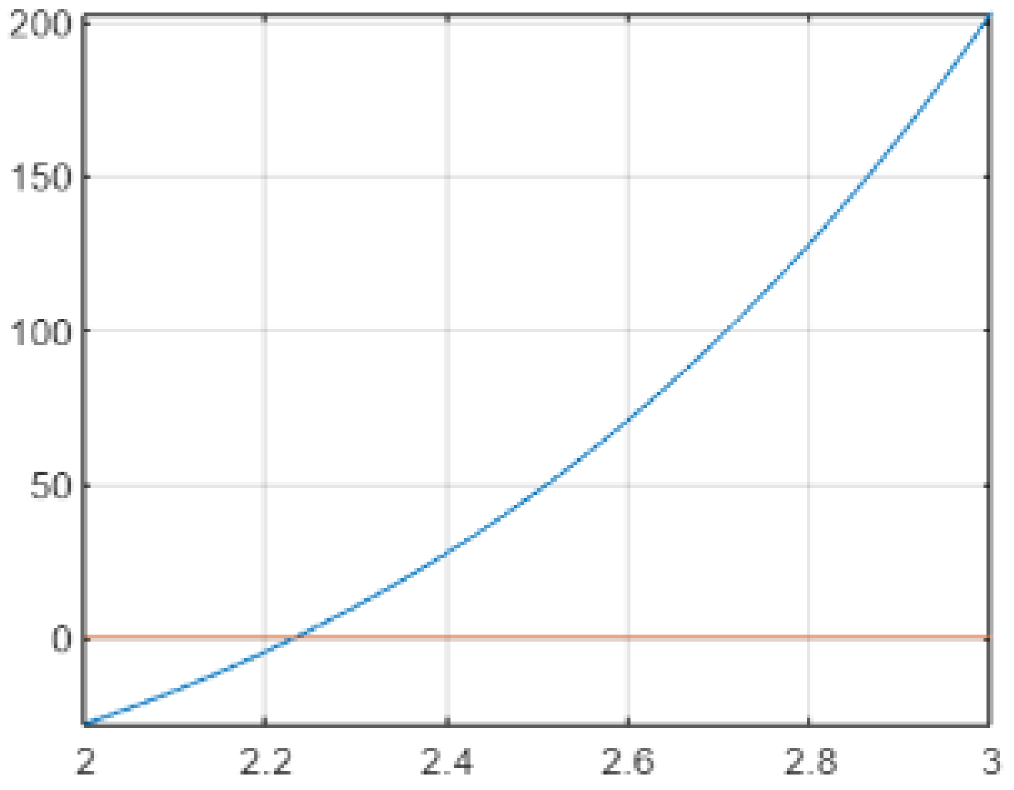
```
fplot({f,0},[-2,-1]), grid
```



# Choice of intervals

So the only root is on [2,3] and we can plot the graph over this interval

```
fplot({f,0},[2,3]), grid
```




**Because f(x)= x^5 + 20*x -100 is a polynomial, the command roots will ensure our answer:**

```
vpa(roots([1,0,0,0,20,-100]), 10)
```

2.231756715

# PRACTICE

$$x + \log(x) = 5$$

$$x^3 + e^{5x} + 1 = 0$$

$$3x + 5 = \cos(x)$$

# ANSWERS:

$$x + \log(x) = 5$$

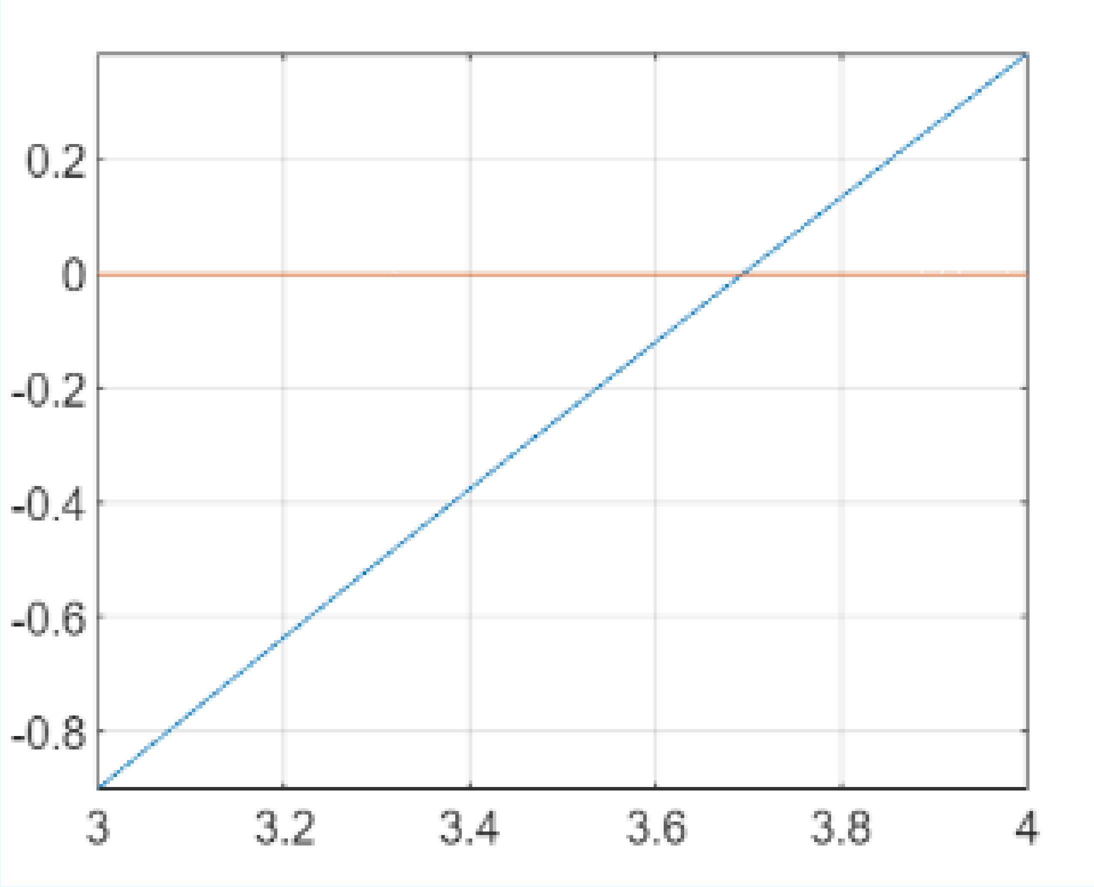
$$x^3 + e^{5x} + 1 = 0$$

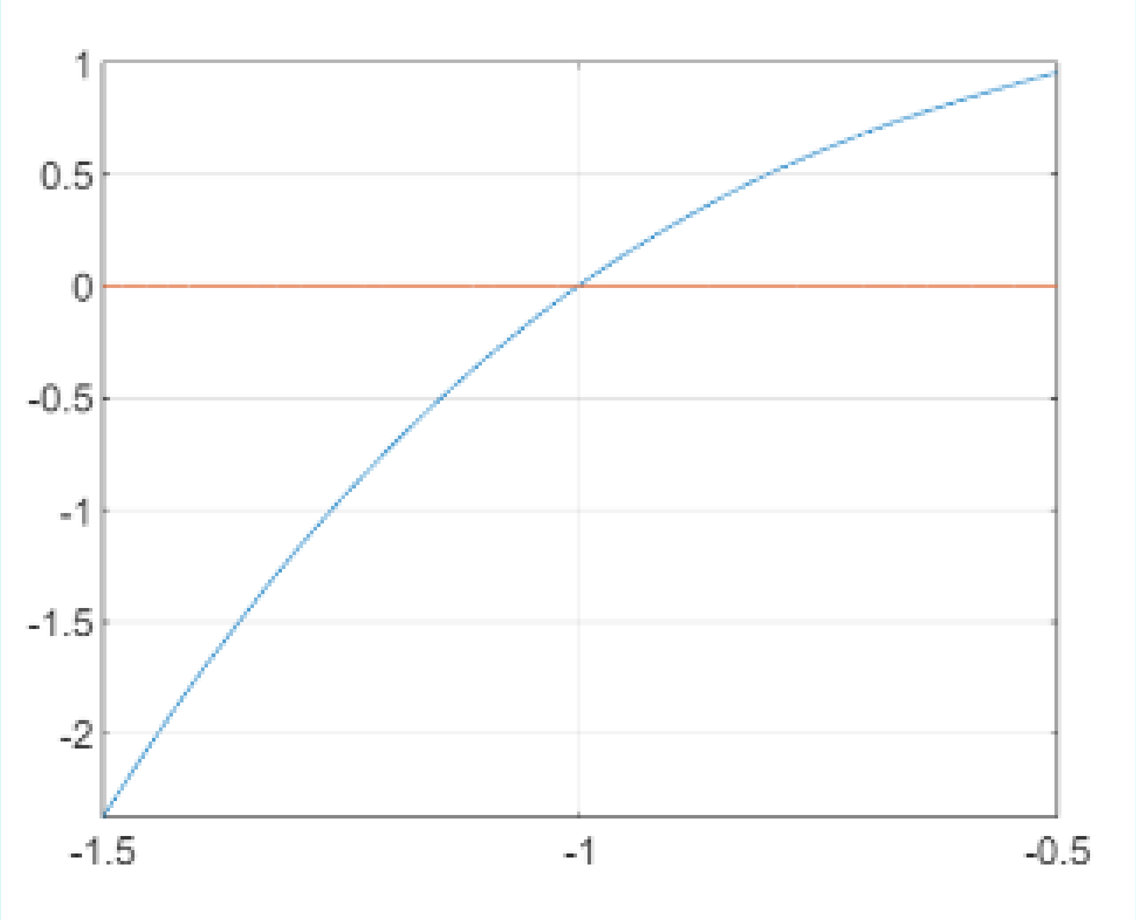
$$3x + 5 = \cos(x)$$

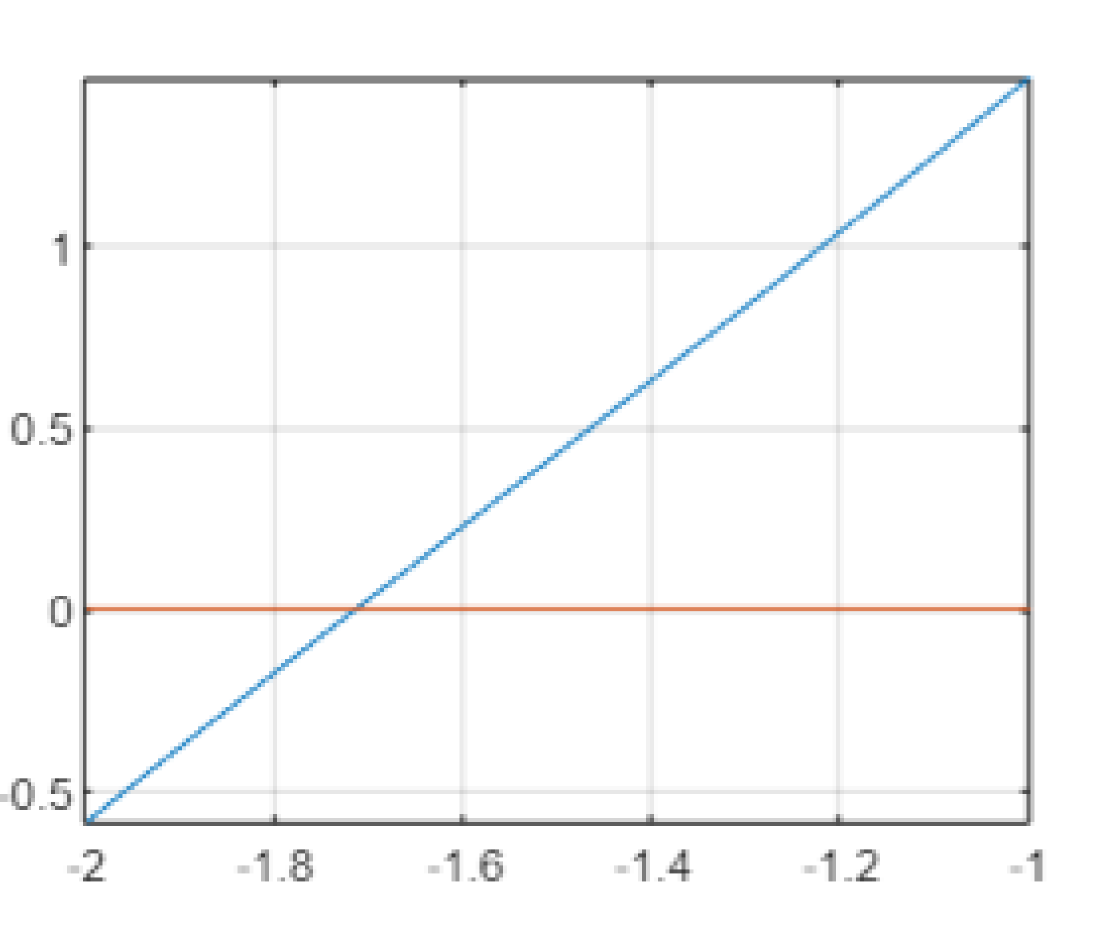
```
fplot({f,0},[3,4]),grid
```

```
fplot({f,0},[-1.5,-0.5]),grid
```

```
fplot({f,0},[-2,-1]), grid
```



# Bisection method

To solve the equation $f(x) = 0$ on [a,b] $(f(a) \cdot f(b) < 0)$ we use the sequence:

$$a_1 = a, \quad b_1 = b, \quad p_1 = \frac{a+b}{2}, \quad i = 1$$

$$\left.\begin{array}{l} \textit{If } f(a_i)f(p_i) < 0 \Rightarrow a_{i+1} = a_i,\ b_{i+1} = p_i \\ \textit{If } f(p_i)f(b_i) < 0 \Rightarrow a_{i+1} = p_i,\ b_{i+1} = b_i \end{array}\right\}$$

$$p_{i+1} = \frac{a_{i+1} + b_{i+1}}{2}$$

**Theorem**

Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) \cdot f(b) < 0$. The bisection method generates a sequence $(p_n)$ that approximates the root $p$ of $f$ in such a way that

$$|p_n - p| \leq \frac{b-a}{2^n}, \quad n \geq 1 \ .$$

**Note:** it is important to know previously that the root on $[a, b]$ is unique.

# Bisection code

Open the file bisection.m

```
function p = bisection(f,a,b,tol)
%
%
%Write comments for this function
%
%
u=f(a);v=f(b);
if sign(u)==sign(v)
    disp('We cannot use the method: sign(f(a))=sign(f(b)')
    return
end
fprintf('Results obtained with bisection method \n')
fprintf('\n')
n=1;
p=(a+b)/2;w=f(p);
fprintf('n=%i a=%3.8f b=%3.8f p=%3.8f \n',n,a,b,p)
while ((b-a)/2>tol)
    if sign(u)==sign(w)
        a=p;u=w;
    else
        b=p;v=w;
    end
    n=n+1;
    p=(a+b)/2;w=f(p);
    fprintf('n=%i a=%3.8f b=%3.8f p=%3.8f \n',n,a,b,p)
end

end
```

# EXAMPLE:

**To get an approximation of the solution of x^5 + 20*x -100=0 on [2,3] up to an error of 10^-6:**

```
syms f(x)
f(x)= x^5 + 20*x -100
f(2)*f(3)<0
p=bisection(f,2,3,10^-6)
vpa(p, 10)
```

```
Results obtained with bisection method

n=1 a=2.00000000 b=3.00000000 p=2.50000000
n=2 a=2.00000000 b=2.50000000 p=2.25000000
n=3 a=2.00000000 b=2.25000000 p=2.12500000
n=4 a=2.12500000 b=2.25000000 p=2.18750000
n=5 a=2.18750000 b=2.25000000 p=2.21875000
n=6 a=2.21875000 b=2.25000000 p=2.23437500
n=7 a=2.21875000 b=2.23437500 p=2.22656250
n=8 a=2.22656250 b=2.23437500 p=2.23046875
n=9 a=2.23046875 b=2.23437500 p=2.23242188
n=10 a=2.23046875 b=2.23242188 p=2.23144531
n=11 a=2.23144531 b=2.23242188 p=2.23193359
n=12 a=2.23144531 b=2.23193359 p=2.23168945
n=13 a=2.23168945 b=2.23193359 p=2.23181152
n=14 a=2.23168945 b=2.23181152 p=2.23175049
n=15 a=2.23175049 b=2.23181152 p=2.23178101
n=16 a=2.23175049 b=2.23178101 p=2.23176575
n=17 a=2.23175049 b=2.23176575 p=2.23175812
n=18 a=2.23175049 b=2.23175812 p=2.23175430
n=19 a=2.23175430 b=2.23175812 p=2.23175621
n=20 a=2.23175621 b=2.23175812 p=2.23175716
```

```
p = 2.2318

ans = 2.231757164
```

# PRACTICE

$$x + \log(x) = 5$$

interval:[3,4]

$$x^3 + e^{5x} + 1 = 0$$

interval:[-1.5,0.5]

$$3x + 5 = \cos(x)$$

interval:[-2,-1]

# ANSWERS:

$x + \log(x) = 5$

```
syms f(x)
f(x)=x+log(x)-5
f(3)*f(4)>0
p=bisection(f,3,4,10^-6)
vpa(p,10)
```

ans = $3.693440437$

$x^3 + e^{5x} + 1 = 0$

```
syms f(x)
f(x)= x^3 + exp(5*x) + 1
f(-1.5) * f(-0.5) <0
p=bisection(f,-1.5,-0.5,10^-6)
vpa(p, 10)
```

ans = $-1.002215385$

$3x + 5 = \cos(x)$

```
syms f(x)
f(x)= 3*x+5-cos(x)
f(-2)*f(-1)<0
p=bisection(f,-2,-1,10^-6)
vpa(p, 10)
```

ans = $-1.714356422$