Calculus

Numbers

Induction

↳ P(1) is true

↳ $P(n) = (n+1)$

$$\underbrace{1+2+3+\dots+n = \frac{n(n+1)}{2}}_{C}$$

1/ $1+2+3+\dots+n = \frac{n(n+1)}{2}$

1st/ Prove the equality for $n=1$

a) $n=1$ b) $\frac{1\cdot(1+1)}{2} = 1$ $a=b$

2nd/ Prove that $P(n) \Rightarrow P(n+1)$

$$\underbrace{1+2+3+\dots+n}_{D} + (n+1) = \frac{(n+1)(n+2)}{2}$$

$$\frac{n(n+1)+2(n+1)}{\cancel{2}} = \frac{(n+1)(n+2)}{\cancel{2}};\quad n^2+n+2n+2 \Leftrightarrow n^2+2n+n+2$$

2/ $\sum_{k=1}^{n} \frac{k}{k!(k+1)} = 1 - \frac{1}{(n+1)!}$

$k=1$ $\frac{1}{1\cdot 2}$ $k=2$ $\frac{2}{2\cdot 1\cdot 3}$ $k=3$ $\frac{3}{3\cdot 2\cdot 1\cdot 4}$

$$P(n) \to \frac{1}{1\cdot 2} + \frac{2}{2\cdot1\cdot3} + \frac{3}{3\cdot2\cdot1\cdot4} + \dots + \frac{n}{n!(n+1)} = \underbrace{1 - \frac{1}{(n+1)!}}_{C}$$

1st/ Prove for $P(n)=1$

a) $\frac{1}{2}$ b) $1-\frac{1}{2} = 1/2$ $a=b$ $C=D$

2nd/ Prove that $P(n) \Rightarrow P(n+1)$

$$\underbrace{\frac{1}{1\cdot 2} + \frac{2}{213} + \frac{3}{3214} + \dots + \frac{n}{n!(n+1)}}_{D} + \frac{(n+1)}{(n+1)!(n+2)} = 1 - \frac{1}{(n+2)!} \Leftrightarrow \cancel{1} - \frac{1}{(n+1)!} + \frac{(n+1)}{(n+1)(n+2)} = \cancel{1} - \frac{1}{(n+2)!} \Leftrightarrow$$

$$\frac{(n+2) \cancel{+} n+1)}{(n+1)(n+2)}$$

Polinomio grado 3 $\quad f(x)=\sqrt{1+x} \quad$ en $x=0$

$$P_3(x)= f(a)+f'(a)\cdot(x-a)+\frac{f''(a)}{2!}\cdot(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\frac{f^{IV}(a)}{4!}(x-a)^4$$

$$P_3(x)= \sqrt{1+0}+\frac{1}{2}\cdot(x-0)^{-1/2}-\frac{1}{4}\left(\frac{1}{2!}\right)^{-3/2}(x-0)^2+\frac{3}{8}\frac{(1)^{-5/2}}{3!}(1-0)^3+\frac{15}{16}\frac{(1)^{-7/2}}{4!}(x-0)^4$$

$$= 1+\frac{x}{2}-\frac{1/4}{2!}x^2+\frac{3/8}{3!}x^3 = 1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3$$

2) $f(x)=\cos x \qquad n=4$

a) Maclaurin polynomial

$$P_5(x)= f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\frac{f^{IV}(a)}{4!}(x-a)^4+\frac{f^{V}(a)}{5!}(x-a)^5+$$
$$+\frac{f^6(a)}{6!}(x-a)^6=$$

$$P_5(x)= \cos 0-\operatorname{sen}(0).(x-0)+\frac{-\cos(0)}{2!}(x-0)^2+\frac{\operatorname{sen}(0)}{3!}(x-0)^3+\frac{\cos(0)}{4!}(x-0)^4+\frac{-\operatorname{sen}(0)}{5!}$$
$$(x-0)^5$$

$$1-\frac{x^2}{2!}+\frac{x^4}{4!}$$

b) Approx of $\cos(-0'13)$

$$1-\frac{(\cos(-0'13))^2}{2!}+\frac{((\cos(-0'13))^4}{4!}=0'9553375\ldots \qquad -3<c<0$$

$$\text{Error}=\left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right|=\frac{f^6(c)}{(n+1)!}(-0'13)^6\Big|=\left|\frac{-\cos(c)}{6}.(0'13)^6\right|\leq\frac{0'13^6}{6!}=1'01\cdot10^6$$

Polinomio de Taylor

$$P_n(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

1/ $f(x) = e^x \quad x=0$

4 primeros

$$P_1(x) = f(0) + f'(0)(x-0) = 1 + 1(x+0) = 1+x$$

$$P_2(x) = P_1(x) + \frac{f''(a)}{2!}(x-0)^2 = 1+x+\frac{x^2}{2}$$

$$P_3(x) = P_2 + \frac{f'''(a)}{3!}(x-0)^3 = 1+x+\frac{x^2}{2}+\frac{x^3}{6}$$

$$P_4(x) = P_3 + \frac{f''''(a)}{4!}(x-0)^4 = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}$$

$$P_n(x) = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\dots+\frac{x^n}{n!}$$

2/ $f(x) = \sin x$ at $a=0$

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^n(a)}{n!}(x-a)^n$$

$$P_n(x) = \text{sen}0 + \cos(0)(x-0) + \frac{-\text{sen}(0)}{2!}(x-0)^2 + \frac{-\cos 0}{3!}(x-0)^3$$

$$P_n(x) = 0 + x + 0 - \frac{x^3}{6}$$

3/ Polinomio de Taylor de orden 3; $f(x) = \text{sen}\,x$ donde $x = \pi/6$ para aproximar sen 1/2

$$P_n(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{IV}(a)}{4!}(x-a)$$

$$P_n(x) = \text{sen}\,\pi/6 + \cos(\pi/6)\cdot(\overset{1/2}{x}-\pi/6) + \frac{-\text{sen}(\pi/6)}{2!}\cdot(\overset{1/2}{x}-\pi/6)^2 + \frac{-\cos(\pi/6)}{3!}\cdot(\overset{1/2}{x}-\pi/6)^3 + \frac{\text{sen}\,\pi/6}{4!}(\overset{1/2}{x}-$$

Para aproximar a 1/6, x por 1/2

$$P_3(1/2) = 0'4794255387$$

$$\text{Error}_3(\pi/6) = \frac{f^{(n+1)}(a)}{(n+1)!}\cdot(t-a)^{n+1} \text{ con } t \in (a,x) \overset{(1/2,\ \pi/6)}{} \quad < t <$$

$$\text{Error}_3(\pi/6) = \frac{f^{IV}(\pi/6)}{4!}(\overset{1/2}{t}-\pi/6)^4 = 6'4613\cdot 10^{-4}$$

Calculus and Numerical Methods. Problem-solving test 3. 7 Dec 2022

Full name: Javier García Tercero

Problem 1. Find the area of the region bounded by the graphs of the functions

$$f(x) = \frac{40}{36+x^2} \qquad \text{and} \qquad g(x) = x^2 - 3\,.$$

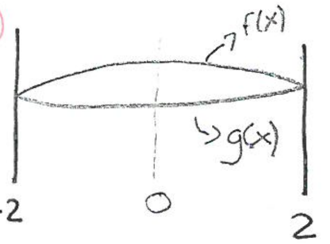
6/9

1/ Itineration points $\quad 2 \quad -2$

1 $\quad f(x) = g(x)\ ;\ \frac{40}{36+x^2} = x^2 - 3 \Rightarrow 40 = (x^2-3)(36+x^2);\ 40 = 36x^2 + x^4 - 108 - 3x^2;$

$x^4 + 33x^2 - 148 = 0 \qquad t = x^2 \qquad t^2 + 33t - 148 = 0 \qquad \frac{-33 \pm \sqrt{33^2 - 4\cdot 1\cdot(-148)}}{2} \begin{cases} 4 = x^2 \\ -37 = x^2 \end{cases}$

0'5



$f(0) = 10/9 = 1'11\ldots$

$g(0) = -3$

$\boxed{f(x) > g(x)\ \forall x \in [-2,2]}$ $\quad x = \sqrt{4} = \pm 2$

$x = \sqrt{-37} = \pm\sqrt{37}\,i \Rightarrow$ NO

$$\int_a^b f(x) - g(x)$$

0'5

$$\int_{-2}^{2} \frac{40}{36+x^2} - x^2 + 3^{*} = \left[\frac{10}{9}\cdot \text{Arctg}\left(\frac{x}{6}\right) - \frac{x^3}{3} + 3x\right]_{-2}^{2}$$

1

$$\left[\frac{10}{9}\cdot{}^{\cdot 1/6}\ \text{Arctg}\left(\frac{2}{6}\right) - \frac{2^3}{3} + 3\cdot 2\right] - \left[\frac{10}{9}\cdot{}^{\cdot 1/6}\ \text{Arctg}\left(\frac{-2}{6}\right) - \frac{(-2)^3}{3} + 3\cdot(-2)\right] = \boxed{10'95\,u^2}$$

Primitives *

$$\int \frac{40}{36+x^2} \Leftrightarrow 40 \int \frac{1/36}{36/36 + \frac{x^2}{36}} \Leftrightarrow \frac{40}{36}\cdot{}^{1/6} \int \frac{1}{1+\left(\frac{x}{6}\right)^2} \Leftrightarrow \frac{10}{9}\cdot{}^{\cdot 1/6}\ \text{Arctg}\left(\frac{x}{6}\right) + C$$

$-2'5$

$$\int x^2 - 3 = \frac{x^3}{3} - 3x + C$$

Problem 2. Find the area of the region bounded by the graph of the function $f(x) = e^{-x}\sin(x)$ and the horizontal axis between $x = -\pi$ and $x = \pi$. Hint: Use that the sine function takes positive values on $(0, \pi)$ and negative values on $(-\pi, 0)$.

$f(x) = e^{-x} \sin(x)$ $\qquad \int u\,dv = uv - \int v\,du$ $\qquad$ Alpes

$u = e^{-x}$ $\qquad du = -e^{-x}$

$dv = \sin x$ $\qquad v = -\cos x$

2 $I = \int e^{-x} \sin x = -e^{-x} \cdot \cos x - \int e^{-x} \cos x \Rightarrow$

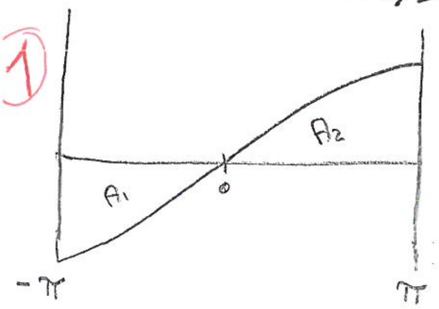
$u = e^{-x}$ $\qquad du = -e^{-x}$

$dv = \cos x$ $\qquad v = \sin x$

$\Rightarrow -e^{-x} \cdot \cos x - \left(e^{-x} \sin x - \int -e^{-x} \sin x \Rightarrow -e^{-x} \cdot \cos x - e^{-x} \sin x + \int -e^{-x} \sin x \Rightarrow\right.$

$\Rightarrow -e^{-x}(\cos x + \sin x) \Big/ -e^{-x} \sin x$ $\qquad A_T = A_2 + A_1$

1



$A_2 = \left[-e^{-x}(\cos x + \sin x)\right]_{-\pi}^{0} = 12'07$

2

$A_1 = \left[-e^{-x} \cdot (\cos x + \sin x)\right]_{0}^{\pi} = -0'521\ldots$

$\boxed{A_T = 12'07 + (-0'521) = 11'549\ u^2}$

$12'07 - (-0'521) = 12'591$

Calculus and Numerical Methods. Problem-solving activity 1. 28 September 2022

Full name: Javier García Tercero

Problem (8 points). Prove by induction: $\displaystyle\sum_{k=1}^{n} \frac{2^k(k-1)}{k(k+1)} = \frac{2^{n+1}}{n+1} - 2, \quad \forall n \in \mathbb{N}.$

8+0'5?

1- Answers for constant "k"

When k = 1 $\qquad$ When k = 2 $\qquad$ When k = 3

$$\frac{2^1\cdot(1-1)}{1(1+1)} = \frac{0}{1(1+1)} = \frac{0}{1\cdot 2} \qquad \frac{2^2(2-1)}{2\cdot(2+1)} = \frac{2^2\cdot 1}{2\cdot 3} \qquad \frac{2^3(3-1)}{3\cdot(3+1)} = \frac{2^3\cdot 2}{3\cdot 4}$$

Principle of induction

$$P(n): \frac{0}{1\cdot 2} + \frac{2^2\cdot 1}{2\cdot 3} + \frac{2^3\cdot 2}{3\cdot 4} + \dots + \frac{2^n(n-1)}{n(n+1)} = \underbrace{\frac{2^{n+1}}{n+1} - 2}_{C}$$

P(n) correctly 1

1st/ Prove the principle for $n = 1$; Leftside must be the same as rightside

a) $\frac{2^1\cdot(1-1)}{1\cdot(1+1)} = \frac{0}{2} = 0$ $\qquad$ b) $\frac{2^{1+1}}{1+1} - 2 = 2 - 2 = 0$; $\qquad$ $a = b$

Holds for n = 1 $\quad$ 1

2nd/ Prove the principle for $n = n+1$

Relationship 1'5

"C" and "D" are the same, so we sustitute it

$$\underbrace{\underbrace{\frac{0}{2} + \frac{2^2\cdot 1}{2\cdot 3} + \frac{2^3\cdot 2}{3\cdot 4} + \frac{2^n(n-1)}{n(n+1)}}_{D} + \frac{2^{n+1}(n)}{(n+1)(n+2)}} = \underbrace{\frac{2^{n+2}}{n+2} - 2} \iff$$

We have to prove that A = B

$$\frac{2^{n+1}}{n+1} - \cancel{2} + \frac{2^{n+1}\cdot n}{(n+1)(n+2)} \overset{A}{=} \frac{2^{n+2}}{n+2} - \cancel{2} \iff \quad B$$

No calculations mistakes

$$2^{n+1}\cdot(n+2) + 2^{n+1}\cdot n = (2^{n+2})(n+1) \iff$$

$$2^{n+1}n + 4^{n+1} + 2^{n+1}n = 2^{n+2}n + 2^{n+2} \iff$$

4

$4^{n+1}n + 4^{n+1} = 2^{n+2}n + 2^{n+2} \iff$

$2^{n+2}n + 2^{n+2} = 2^{n+2}n + 2^{n+2} \iff$

Left side is the same as the right side so the principle has been proved for this problem

Conclusion ✓

$2^{n+1}n + 2^{n+1}n = \underline{4^{n+1}n}$

$2 \cdot 2^{n+1}n = 2^{n+2}n$

Question (1 point). For each $n \in \mathbb{N}$, we call $P(n)$ the formula

$$(3n) + (3n+3) + (3n+6) + \cdots + (6n) = \frac{9n(n+1)}{2}$$

Write the formula $P(n+1)$.

Note: there must be a clear connection between $P(n)$ and what you write for $P(n+1)$.

Formula for $P(n+1)$

0'5?

$(\cancel{3n}) + (3n+3) + (3n+6) + \dots + (6n) + (6n+6) + (6n+3)^* = \frac{9(n+1)\cdot(n+2)}{2}$

No

*$(6n+3)$ is the missing term to prove the formula $P(n+1)$; it cant be $(6n+1)$, $(6n+2)$... because they are not part of the solution

Problem 2. (4.5 points) We define $f(x) = \sin(x)$.

1'5 a) Find approximations of the number $\sin(-0.4)$ (that is, -0.4 radians) given by the Maclaurin polynomials of f of orders 5 and 6. $a=0$ $X=-0'4$

1'5 b) Determine an upper bound of each error in part a). Are these upper bounds really different?

0 c) Determine the order of the Maclaurin polynomial of f that can be used to find an approximation of $\sin(15^o)$, that is 15 degrees, up to an error of 10^{-5}.

$$a)\ P_n(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}\cdot(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

$$P_6(x) = \sin(x) + \cos(x)\cdot(x-a) + \frac{-\sin(x)}{2!}(x-a)^2 + \frac{-\cos(x)}{3!}(x-a)^3 + \frac{\sin(x)}{4!}\cdot(x-a)^4 + \frac{\cos(x)}{5!}(x-a)^5 + \frac{-\sin(x)}{6!}(x-a)^6$$

1- Maclaurin polynomial of order 5 to $\sin(-0'4)$ (✱)

$$P_5(x) = \cancel{\sin(0)} + \cos(0)\cdot(-0'4-0) - \cancel{\frac{\sin(0)}{2!}}(-0'4-0)^2 - \frac{\cos(0)}{3!}(-0'4-0)^3 + \cancel{\frac{\sin(0)}{4!}}(-0'4-0)^4 + \frac{\cos(0)}{5!}(-0'4-0)^5$$

$$P_5(x) = -0'4 - \frac{(-0'4)^3}{6} + \frac{(-0'4)^5}{120} = -0'3894186666$$ ✓

2- Maclaurin polynomial of order 6 to $\sin(-0'4)$

$$P_6(x) = P_5(x) + \frac{f^{vi}(a)}{6!}(x-a)^6 = x - \frac{x^3}{6} + \frac{x^5}{120} + \frac{-\cancel{\sin(0)}}{6!}(x-a)^6 = x - \frac{x^3}{6} + \frac{x^5}{120} = -0'3894186666$$ ✓

b) $$\text{Error}_n = \left|\frac{f^{(n+1)}(c)}{(n+1)!}\cdot(x-a)^{n+1}\right|$$ because max cos: ±1 & sin = ±1 $\quad x < c < a$

$$\text{Error}_5 = \left|\frac{f^{(6)}(c)}{6!}(x-a)^6\right| \le \frac{1}{6!}(x-a)^6 \Rightarrow \left|\frac{1}{6!}\cdot(-0'4-0)^6\right| = 5'688\cdot10^{-6}$$ ✓

$$\text{Error}_6 = \left|\frac{f^{(7)}(c)}{7!}(x-a)^7\right| \le \frac{1}{7!}(x-a)^7 \Rightarrow \left|\frac{1}{7!}\cdot(-0'4-0)^7\right| = 3'250\cdot10^{-7}$$ ✓

It is a big difference

c) Made on the other side

(✱) $$P_5(x) = \sin(0) + \cos(0)\cdot(x-0) - \frac{\sin(0)}{2!}(x-0)^2 - \frac{\cos(0)}{3!}(x-0)^3 + \frac{\sin(0)}{4!}(x-0)^4 + \frac{\cos(0)}{5!}(x-0)^5$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120}$$ ✓

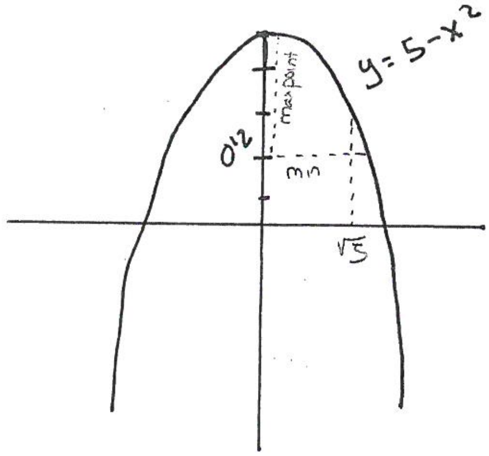
Calculus and Numerical Methods. Problem solving activity 2. 15 Nov 2022

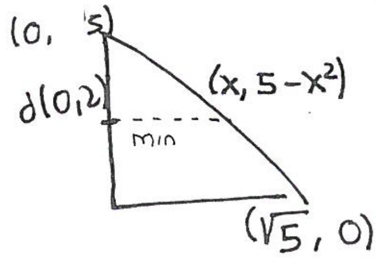
$3+1=4$

Full name: Javier García Tercero

Problem 1. (4.5 points) Find the points of the parabola $y = 5 - x^2$ between $(0, 5)$ and $(\sqrt{5}, 0)$ whose distance to the point $(0, 2)$ is a maximum or a minimum. What are the maximum and minimum distances from these points to the parabola?

0





$f(x) = -x^2+5-2 = 0; -x^2+3$

$f'(x) = -2x = 0; x=0$

$f''(x) = -2$

Max point to d(0,2) to the parabola is (0,5)

0 c) $15 \cdot \frac{\pi}{2} = 15\pi/2$ rad

$x < c < a$

$$Error_n = \left| \frac{f^{(n+1)}(c)}{(n+1)!} \cdot (x-a)^{n+1} \right| < 10^{-5} \quad ; n=4$$

7

Trapezoidal rule (simple)/Trapezoid rule

$$I = \int_a^b f(x)dx \simeq I^* = \frac{b-a}{2}(f(a)+f(b))$$

If f has a continuous second derivative on $[a,b]$, and $|f''(x)| \leq M_2$, $\forall x \in [a,b]$, then

$$|I - I^*| \leq \frac{M_2}{12}(b-a)^3$$

Remark. The approximation given is exact if the function f is a one degree polynomial.

Composite trapezoid rule

For $n \in \mathbb{N}$, we evaluate the function f at points $x_k = a + k \cdot h$, $k = 0,1,2,\ldots,n$ where $h = \dfrac{b-a}{n}$ (the interval is divided into n equal subintervals).

$$I = \int_a^b f(x)\,dx \simeq I^* = \frac{b-a}{2n}\left(f(a) + 2\sum_{k=1}^{n-1} f(a+kh) + f(b)\right)$$

If f has a continuous second derivative on $[a,b]$ and $|f''(x)| \leq M_2$, $\forall x \in [a,b]$, then

$$|I - I^*| \leq \frac{1}{12}\frac{(b-a)^3}{n^2}M_2 \,.$$

FORMULAS EXAM

Using the trapezoid rule for an error given

To approximate $\int_a^b f(x)\,dx$ up to an error tol using the trapezoid rule, we proceed as follows:

1. Find a bound M_2 such that $|f''(x)| \leqslant M_2$, $\forall x \in [a,b]$.
2. Find n such that $\dfrac{M_2(b-a)^3}{12n^2} < tol$ and determine $h = \dfrac{b-a}{n}$, the number of subdivisions of $[a,b]$.
3. Determine the points $x_k = a + k \cdot h$, $k = 0,1,\ldots,n$ and the values $f(x_k)$.
4. Find the approximation

$$I^* = \int_a^b f(x)dx \simeq \frac{h}{2}\left(f(a) + 2\sum_{k=1}^{n-1} f(x_k) + f(b)\right)$$

Simpson's rule (simple)

We evaluate a function f at $x = a$, $x = \frac{a+b}{2}$ and $x = b$, and approximate

$$I = \int_a^b f(x)\,dx \simeq I^* = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

If f has a continuous fourth derivative on $[a,b]$, then

$$|I - I^*| \leq \frac{M_4}{90}\left(\frac{b-a}{2}\right)^5$$

where

$$| f^{(4)}(x) | \leq M_4 \quad \forall x \in [a,b]$$

Remark. Simpson's rule gives exact results for polynomials up to third degree.

Composite Simpson's rule

For $n \in \mathbb{N}$, we evaluate f at the $2n+1$ points

$$x_k = a + k \cdot h, \quad \text{where} \quad h = \frac{b-a}{2n} \quad \text{and} \quad k = 0,1,\ldots,2n$$

(the interval is divided into $2n$ equal subintervals). Then

$$I = \int_a^b f(x)\,dx \simeq I^* = \frac{b-a}{6n}\left(f(a) + 2\sum_{k=1}^{n-1} f(x_{2k}) + 4\sum_{k=1}^{n} f(x_{2k-1}) + f(b)\right)$$

If $| f^{(4)}(x) | \leq M_4$ $\forall x \in [a,b]$, then

$$|I - I^*| \leq \frac{M_4(b-a)^5}{90 \cdot 2^5 \cdot n^4}$$

Using Simpson's rule with a given error

To approximate $\int_a^b f(x)\,dx$ up to an error tol using Simpson's rule, we proceed as follows:

1. Find M_4 such that $\left|f^{(iv)}(x)\right| \leqslant M_4$, $x \in [a,b]$.
2. Find n such that $\dfrac{M_4(b-a)^5}{2^5 \cdot 90 \cdot n^4} < tol$ and determine $h = \dfrac{b-a}{2n}$.
3. Determine the points $x_j = a + j \cdot h$, for $j = 0,1,\ldots,2n$ and the values $f(x_j)$.
4. Find the approximation

$$\int_a^b f(x)dx \simeq \frac{b-a}{6n}\left(f(a) + 4\sum_{j=1}^{n} f(x_{2j-1}) + 2\sum_{j=1}^{n-1} f(x_{2j}) + f(b)\right)$$

2) $\lim\limits_{n\to\infty} \dfrac{\log(1^2)+\log(2^2)+\dots+\log(n^2)}{\sqrt{n}}$ $\hookrightarrow b_n = +\infty$

Stoltz $\lim\limits_{n\to\infty} = \dfrac{a_{n+1}-a_n}{b_{n+1}-b_n}$

$$\lim_{n\to\infty} \frac{\log(1^2)+\log(2^2)+\dots+\log(n^2)+\log(n+1)^2-\log(1^2)+\log(2^2)+\dots+\log(n^2)}{\sqrt{n+1}-\sqrt{n}} =$$

$$= \lim_{n\to\infty} \frac{\log(n+1)^2}{\sqrt{n+1}-\sqrt{n}} = \frac{\infty}{\infty} \to \text{L'hopital} = \frac{\frac{2n+2}{n^2+2n+1}}{\frac{(n+1)^{1/2}-n^{1/2}}{1}} = \frac{(n^2+2n+1)\cdot(n+1)^{1/2}-n^{1/2}}{2n+2}$$

$=2$

Sandwich theoreme

a) $\lim\limits_{x\to 0} x^2\cos\left(\frac{1}{x}\right)$

$\hookrightarrow$

$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$

$-x^2 \leq x^2\cos\left(\frac{1}{x}\right) \leq x^2$

$\lim\limits_{x\to 0}$ $-x^2 \leq \lim\limits_{x\to 0} x^2\cdot\cos\left(\frac{1}{x}\right) \leq \lim\limits_{x\to 0} x^2$

$0 \leq \lim\limits_{x\to 0} x^2\cdot\cos\left(\frac{1}{x}\right) \leq 0$

b) Prove that $\lim\limits_{x\to 0^+} \sqrt{x}\ e^{\sin(\pi/x)} = 0$

$0 \leq e^{\sin(\pi/x)} \leq 2'71$; $0\cdot\sqrt{x} \leq e^{\sin(\pi/x)} \leq 2'71\cdot\sqrt{x}$

$0 \leq \lim\limits_{x\to 0^+} e^{\sin(\pi/x)} \leq \lim\limits_{x\to 0^+} 2'71\cdot\sqrt{x}$; $x=0$

1) $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)} = \frac{n}{3(n+3)}$ For $k=1$ $\frac{1}{12}$ For $k=2$ $\frac{1}{20}$

$$\frac{1}{12}+\frac{1}{20}+\dots+\frac{1}{(n+2)(n+3)} = \frac{n}{3(n+3)}$$

i) Prove for $P(n)=1$

$P(n) = \frac{1}{12} = \frac{1}{12}$ ✓

ii) Prove for $P(n+1)$

$$\frac{1}{12}+\frac{1}{20}+\dots+\frac{1}{(n+2)(n+3)}+\frac{1}{(n+3)(n+4)} = \frac{n+1}{3(n+4)};$$

$$\frac{n}{3(n+3)}+\frac{1}{(n+3)(n+4)} = \frac{n+1}{3(n+4)};$$

$$\cancel{\frac{n}{3(n+3)}+\frac{1}{(n+3)(n+4)} = \frac{n+1}{3(n+4)}}$$

$$\cancel{\frac{(n+4)+3}{3(n+3)(n+4)} = \frac{(n+1)(n+3)}{3(n+4)(n+3)} + \frac{1}{(n}}$$

$$\frac{n\cdot(n+4)+3}{3(n+3)(n+4} = \frac{(n+1)(n+3)}{3(n+3)(n+4)};\quad n^2+4n+3 = n^2+3n+n+3$$

$$n^2+4n+3 = n^2+4\cdot n+3$$

1/ By induction

$$\sum_{k=1}^{n} \frac{2k+3}{(k+1)^2(k+2)^2} = \frac{1}{4} - \frac{1}{(n+2)^2}$$

For $k=1$ For $k=2$

$$\frac{5}{2^2 \cdot 3^2} \qquad \frac{7}{3^2 \cdot 4^2}$$

$$P(n): \overbrace{\frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \cdots + \frac{2n+3}{(n+1)^2(n+2)^2}}^{C} = \overbrace{\frac{1}{4} - \frac{1}{(n+2)^2}}^{D}$$

i) Prove for $n=1$

$$\frac{5}{36} = \frac{1}{4} - \frac{1}{3^2} ; \quad \frac{5}{36} = \frac{5}{36}$$

ii) Prove for $n = n+1$

$$P(n+1): \frac{5}{2^2 3^2} + \frac{7}{3^2 \cdot 4^2} + \cdots + \frac{2n+3}{(n+1)^2(n+2)^2} + \frac{2(n+1)+3}{(n+2)^2(n+3)^2} = \frac{1}{4} - \frac{1}{(n+3)^2}$$

$$= \frac{1}{4} - \frac{1}{(n+2)^2} + \frac{2n+5}{(n+2)^2 \cdot (n+3)^2} = \frac{1}{4} - \frac{1}{(n+3)^2} =$$

$$= \frac{(n+2)^2(n+3)^2 - 4(n+3)^2 + 8n+20}{4 \cdot (n+2)^2(n+3)^2} = \frac{(n+2)^2 \cdot (n+3)^2 - (4(n+2)^2)}{4 \cdot (n+2)^2 \cdot (n+3)^2}$$

$$= [n^2+4+4n] \cdot [n^2+9+6n] - (4n+12)^2 + 8n + 20 = (n^2+4+4n)(n^2+9+6n) - (4(n^2+4+4n))$$

$$= n^4 + 9n^2 + 6n^3 + 4n^2 + 36 + 24n + 4n^3 + 36n + 24n^2 - 16n^2 - 144 - 96n + 8n + 20$$

$$n^4 + 10n^3 + 21n^2 - 28n - 88 = n^4 + 9n^2 + 6n^3 + 4n^2 + 36 + 24n + 4n^3 + 36n + 24n^2 - 16n - 4n^2 - 16$$

$$n^4 + 10n^3 + 21n^2 - 28n - 88 = n^4 + 10n^3 + 33n^2 + 44 + 20$$

2) Convergent or divergent

$$\sum_{n=1}^{\infty} \left(\frac{2}{\sqrt{n}} + \left(\frac{n^2+1}{n^2} \right)^n \right) = 1^\infty = e^{L_{n\to\infty}} n \cdot \left(\frac{2n^2 + (n^2+1) \cdot n^{1/2}}{\sqrt{n} + n^2} - 1 \right) =$$

$$\frac{2n^2 + (n^2+1)\sqrt{n}}{\sqrt{n} + n^2} \Bigg\} \quad e^{L_{n\to\infty}} n \cdot \left(\frac{2n^2 + n + n^{1/2} - n^{1/2} - n^2}{\sqrt{n} + n^2} \right) = e^{L_{n\to\infty}} n \left(\frac{n^2+n}{\sqrt{n} + n^2} \right)$$

$$e^{L_{n\to\infty}} \left(\frac{n^3 + n^2}{\sqrt{n} + n^2} \right) = e^{\infty} = +\infty ;$$ Divergent because it doesnt have a specific number as

answer

2) $\cos\left(\frac{x}{10}\right) - 5x = 0$

$f(x) = \cos\left(\frac{x}{10}\right) - 5x$

$f'(x) = -\sin\left(\frac{x}{10}\right)\cdot\frac{1}{10}$

$f''(x) = -\cos\left(\frac{x}{10}\right)\cdot\frac{1}{100}$

Derivative $\cos\left(\frac{x}{10}\right)$

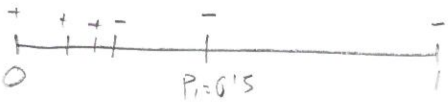
$\frac{10 - x\cdot 0}{10^2} = \frac{-\sin\left(\frac{x}{10}\right)}{10}$

a) Prove 1 root $[a, a+1]$

$f(0) = 1 > 0$ $\quad [0,1]$ $\forall x \in$ One real root

$f(1) = 4 < 0$ $\quad$ Bolzano's th

b) Bisection p up to an error of 0'1



$\frac{b-a}{2^n} < \text{Error};\ \frac{1-0}{2^n} < 0'1;\ \frac{1}{0'1} < 2^n$

$\log 10 < n \log 2;\ \log 10 / \log 2 < n;\ 3'32 < n;\ n = 4$

$P_1 = 0'5$ $\quad f(0'5) = -$

$P_2 = 0'25$ $\quad f(0'25) = -$

$P_3 = 0'125$ $\quad f(0'125) = +$

$P_4 = 0'1875$ $\quad f(0'1875) = +$

$[0'1875, 0'25]$

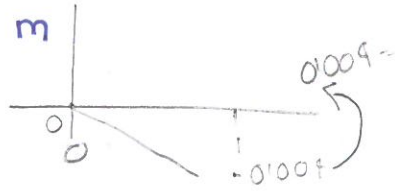
c) Newton's method

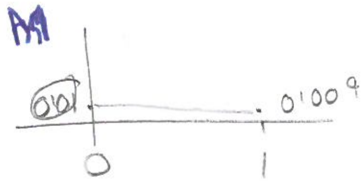
Conditions

1- $f(a)\cdot f(b) < 0 \Rightarrow f(0)\cdot f(1) < 0$

2- Sign $f'(x) < 0$

3- Sign $f''(x) < 0$





p_0 choice

Same sign $\begin{cases} f(p_0) < 0 & 0'25 = p_0 \\ f''(p_0) < 0 & 0'25 \end{cases}$

$P_1 = P_0 - \frac{F(p_0)}{F'(p_0)} = 0'25 - \frac{-0'00249}{-0'0049} = 0'000922$

$P_2 = P_1 - \frac{f(p_1)}{f'(p_1)} = 0'000922 - \frac{-9'21\cdot 10^{-6}}{-0'00999} = 0'001843$

$|P - P_1| \le \frac{M}{2m}\cdot|P_1 - P_0|^2 =$

$= \frac{0'01}{2\cdot 0'009}\cdot \text{ans} = 0'0344$ $\quad 3'44\cdot 10^{-2}$

$= \frac{0'01}{2\cdot 0'009}\cdot \text{ans} = 4'71\cdot 10^{-8}$

Stoltz

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = a \quad \leadsto \quad \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = a$$

Si $b_n \nearrow$ y $\lim_{n\to\infty} b_n = +\infty$

$$\Longrightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = a$$

Si $b_n \searrow$ y $\lim_{n\to\infty} b_n = 0$

a) $$\lim_{n\to\infty} \frac{\sum_{n=1}^{n} n!}{n!} \to \lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \lim_{n\to\infty} \frac{(1!+2!+n!)-(1!+2!+(n-1)!)}{n! - (n-1)!}$$

$n! \hookrightarrow n(n-1)!$

$a_n = \sum_{n=1}^{n} n! = 1!+2!+3!+n!$

$$= \lim_{n\to\infty} \frac{n!}{n(x-1)! - (n-1)!} =$$

$a_{n-1} = \sum_{n=1}^{n} (n-1)! = 1!+2!+3!+(n-1)!$

$$= \lim_{n\to\infty} \frac{n(n-1)!}{n(n-1)! - (n-1)!} =$$

$b_n = n!$

$b_{n-1} = (n-1)!$

$$= \lim_{n\to\infty} = \frac{n}{n-1} = 1$$

b) $$\lim_{n\to\infty} \frac{\sum_{n=1}^{n} \sqrt{n}}{n \cdot \sqrt{n}} = \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \frac{(\sqrt{1}+\sqrt{2}+\sqrt{m}+\sqrt{m+1}) - (\sqrt{1}+\sqrt{2}+\sqrt{n})}{(n+1)\sqrt{n+1} - n\sqrt{n}} =$$

$\sum_{n=1}^{n}\sqrt{n} \to \sqrt{1}+\sqrt{2}+\dots+\sqrt{n}$

$n\cdot\sqrt{n}$: $\sqrt{n^3}$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}\,[(n+1)\sqrt{n+1} + (n\sqrt{n})]}{(n+1)^3 - n^3} = \lim_{n\to\infty} \frac{(n+1)^2 + n\sqrt{n+(n+1)}}{(n+1)^3 - n^3}$$

c) $$\lim_{n\to\infty} \frac{\sqrt{n+1^2} + \dots + \sqrt{n+n^2}}{2n^2} = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$$

$2n^2 \hookrightarrow b_n = +\infty$

$$= \lim_{n\to\infty} \frac{(\sqrt{n+1^2} + \dots + \sqrt{n+n^2} + \sqrt{n+(n+1)^2}) - \sqrt{n+1} + \dots + \sqrt{n+n^2}}{2(n+1)^2 - 2n^2} =$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+(n+1)^2}}{2(n+1)^2 - 2n^2} = \lim_{n\to\infty} \frac{\sqrt{n^2+3n+1}}{4n+2} = 1/4$$

d) $$\lim_{n\to\infty} \frac{1^2+2^2+\dots+n^2}{n^3} ; \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$$

$n^3 \to b_n = \infty$

$$\lim_{n\to\infty} \frac{(1^2+2^2+\dots+n^2+(n+1)^2) - (1^2+2^2+\dots+(n)^2)}{(n+1)^3 - n^3} = \lim_{n\to\infty} \frac{(n+1)^2}{n^3+3n^2+3n+1-3n} = 1/3$$

Sequences

a_n: General term of the sequence

→ Limits

$\lim_{n\to\infty} a_n = l \Rightarrow$ convergent a_n to l

$\lim_{n\to\infty} a_n = +\infty$ a_n divergent to $+\infty$

$\lim_{n\to\infty} a_n = -\infty$ a_n divergent to $-\infty$

Convergent
↓
Real limit

Divergent
↓
to $\pm\infty$
oscillating

Indeterminations

$\infty - \infty$ $\quad \frac{\infty}{\infty}$ $\quad \infty \cdot 0$ $\quad \frac{0}{0}$

Conjugate

$$\lim_{n\to\infty} \sqrt{2n^2+3} - \sqrt{2n^2-n} = \infty - \infty$$

$$\frac{\left(\sqrt{2n^2+3} - \sqrt{2n^2-n}\right)\left(\sqrt{2n^2+3} + \sqrt{2n^2-n}\right)}{\sqrt{2n^2+3} + \sqrt{2n^2-n}} = \frac{\cancel{2n^2}+3-\cancel{2}n^2-n}{\sqrt{2n^2+3}+\sqrt{2n^2-n}} =$$

$$= \frac{3-n}{\sqrt{2n^2+3}+\sqrt{2n^2-n}}$$

Number "e"

$$\lim_{n\to\infty} a_n^{b_n} = e^{\lim_{n\to\infty} b_n (a_n - 1)}$$

a) $$\lim_{n\to\infty} \left(\frac{n^2+1}{n^2-3}\right)^{2n^2+4} = 1^\infty \to e^{\lim_{n\to\infty} 2n^2+4\left(\frac{n^2+1}{n^2-3}-1\right)} =$$

$$= 2n^2+4\left(\frac{n^2+1-n^2+3}{n^2-3}\right) = 2n^2+4\left(\frac{4}{n^2-3}\right) = e^{\lim_{n\to\infty}\left(\frac{8n^2+16}{n^2-3}\right)} =$$

e^8

b) $$\lim_{n\to\infty} \left(\frac{\sqrt{n^2+2n}}{\sqrt{n^2-1}}\right)^n = 1^\infty = e^{\lim_{n\to\infty} n\cdot\left(\frac{\sqrt{n^2+2n}}{\sqrt{n^2-1}}-1\right)} =$$

$$= n\cdot\left(\frac{\sqrt{n^2+2n}-\sqrt{n^2-1}}{\sqrt{n^2-1}}\right) = n\cdot\left(\frac{n^2+2n+n^2-1}{\sqrt{n^2-1}\cdot\sqrt{n^2+2n}+\sqrt{n^2-1}}\right) = e^1$$

$1^3+2^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$

i) Prove for $n=1$

$1^3=\dfrac{1^2(2^2}{4}$; $1=1$

ii) $n=k$ $\quad \overbrace{1^3+2^3+\cdots+k^3}^{C}=\overbrace{\dfrac{k^2(k+1)^2}{4}}^{D}$

$k=k+1$; $1^3+2^3+\cdots+k^3+(k+1)^3=\dfrac{(k+1)^2(k+1)^2}{4}$

$$\frac{k^2(k+1)^2}{4}+(k+1)^3=\frac{(k+1)^2(k+1)^2}{4}$$

$(k+1)^2(k^2+4(k+1)=(k+1)^2(k+2)^2$

$(k+1)^2(k^2+4k+4)$

$(k+1)^2(k+2)(k+2)$

$(k+1)^2(k+2)^2=(k+1)^2(k+2)^2$

c) $\displaystyle\sum_{k=1}^{n} k(k+1)=\frac{n(n+1)(n+2)}{3}$

For $k=1$ $\quad$ For $k=2$ $\quad$ For $k=3$

$1(1+1)=2$ $\quad$ $2\cdot(2+1)=6$ $\quad$ $3(3+1)=12$

$P(n)$ $\quad 2+6+12+\cdots+n(n+1)=\dfrac{n(n+1)(n+2)}{3}$

i) Prove for $n=1$

$1(1+1)=\dfrac{1(1+1)(1+2)}{3}$; $2=2$ ✓

ii) Change n by k

$2+6+12+\cdots+k(k+1)=\overbrace{\dfrac{k(k+1)(k+2)}{3}}^{C}$

$2\cdot k=k+1$

$\overbrace{2+6+12+\cdots+k(k+1)}^{C}+(k+1)(k+2)=\dfrac{(k+1)(k+2)(k+3)}{3}$

$$\frac{k(k+1)(k+2)}{3}+(k+1)(k+2)=\frac{(k+1)(k+2)(k+3)}{3}$$

$k(k+1)(k+2)+3(k^2+2k+k+2)=(k+1)(k+2)(k+3)$

$(k^2+k)(k+2)+3k^2+9k+6=(k^2+2k+k+2)(k+3)$; $k^3+2k^2+k^2+2k+3k^2+9k+6$

$k^3+6k^2+11k+6=k^3+3k^2+3k^2+9k+2k+6$; $k^3+6k^2+11k+6=k^3+6k^2+11k+6$

Induction

formula: $1+2+3+\dots+n = \frac{n(n+1)}{2} \Rightarrow P(n)$

i) P(1) is true

ii) If P(n) = true, then P(n+1) is also true

$P(n): 1+2+3+\dots+n = \frac{n(n+1)}{2}$

i) $P(1): 1 = \frac{1\cdot(1+1)}{2}; \quad 1=1$

ii) If P(n) is true, then P(n+1) is also true

$1+2+3+\dots+n+(n+1) = \frac{(n+1)\,n}{2} + (n+1) = \frac{(n+1)\cdot(n+2)}{2}$

1.) Prove by induction

a) $1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

For $p(1) = 1^2 = \frac{1(1+1)(2+1)}{6}; \quad 1=1$

We add $(n+1)^2$ at both sides

$1^2+2^2+\dots+n^2+(n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n(n+1)(2n+1)+6(n+1)^2}{6} =$

$= \frac{(n+1)(2n^2+7n+6)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$

b) $3+7+11+\dots+(4n-1) = n(2n+1)$

i) Prove for $n=1$

$4\cdot1-1 = 1(2\cdot1+1); \; 3=3$

Sustitute for $n=k$

$\underbrace{3+7+11+\dots+(4k-1)}_{C} = \underbrace{k(2k+1)}_{D}$

$C = D$

$n=k+1$

$\underbrace{3+7+11+\dots+(4k-1)}_{C} + [4(k+1)-1] = (k+1)[2(k+1)+1]$

$k(2k+1) + [4(k+1)-1] = (k+1)[2(k+1)+1];$

$2k^2+k+(4k+4-1) = (k+1)(2k+3\,1);$

$2k^2+k+4k+3 = 2k^2+3k+2k+3;$

$2k^2+5k+3 = 2k^2+5k+3$

3/ Equations

$x^4 + x - 10 = 0$ $f(x) = x^4 + x - 10$ $f'(x) = 4x^3 + 1$ $f''(x) = 12x^2$

a) 1 root in $[0, +\infty)$ that $[a, a+1]$

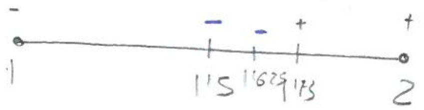
$f'(x) = 4x^3 + 1$; strictly increasing in $[0, \infty)$

$f(1) = -8$ $[1,2]$
$f(2) = 8$

b) Bisection to an error less than 0'2

$$\frac{b-a}{2^n} < \text{Error} \; ; \; \frac{2-1}{2^n} = 0'2 \; ; \; \frac{1}{0'2} < 2^n \; ; \; 5 < 2^n ; \; \log 5 < n \log 2$$

$$\frac{\log 5}{\log 2} < n \; ; \; 2'32 < n ; \; n = 3$$



$[1'625, 1'75]$

$P_1 = 1'5$ $f(1'5) = -$
$P_2 = 1'75$ $f(1'75) = +$
$P_3 = 1'625$ $f(1'625) = -$

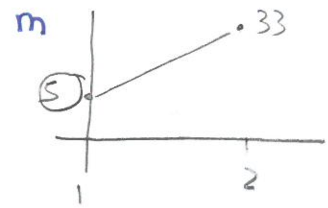
c) Newton p up to an error of 10^{-3}

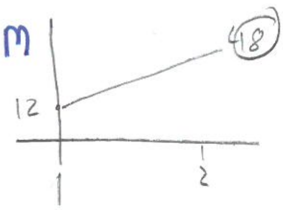
Conditions

1- $f(a) \cdot f(b) < 0 \Rightarrow f(1) \cdot f(2) < 0$

2- Sign of f' = $f'(x) > 0 \quad \forall x \in [0, \infty)$

3- Sign of f'' = $f''(x) \geq 0 \quad \forall x \in [0, \infty)$





* po choice *

$f'(p_0) > 0$ $p_0 = 1'75$
$f''(p_0) > 0$

$$P_1 = P_0 - \frac{f(p_0)}{f'(p_0)} = 1'75 - \frac{1'1284}{22'4373} = 1'64968$$

$$P_2 = P_1 - \frac{f(p_1)}{f'(p_1)} = 1'64968 - \frac{0'4562}{20'16490} = 1'67757$$

$$P_3 = P_2 - \frac{f(p_2)}{f'(p_2)} = 1'67757 - \frac{-0'4024}{19'8843} = 1'69780\ldots$$

$$|p - P_1| \leq \frac{M}{2m} \cdot |P_1 - P_0|^2 = \frac{48}{2 \cdot 5} \cdot |1'64968 - 1'75|^2 = 0'0121\ldots \simeq 1'2 \cdot 10^{-2}$$

$$|p - P_2| \leq \frac{M}{2m} \cdot |P_2 - P_1|^2 = \frac{48}{10} \cdot |1'67757 - 1'64968|^2 = 0'0023 \ldots \simeq 2'3 \cdot 10^{-3}$$

$$|p - P_3| \leq \frac{M}{2m} \cdot |P_3 - P_2|^2 = 4'8 \cdot |1'69780 - 1'67757|^2 = 0'001 \simeq \underline{1 \cdot 10^{-3}}$$

U2_Jun 2021

1/ $f(x) = e^x$

a) Approx of $e^{-0'38}$ 4 order Maclaurin $x = -0'38$ $a = 0$

$$P(x) = e^x + e^x(x-a) + \frac{e^x}{2!}(x-a)^2 + \frac{e^x}{3!}(x-a)^3 + \frac{e^x}{4!}(x-a)^4$$

$$P(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$P(-0'38) = 0'70221914$

$-0'38 < c < 0$

c increasing

b) Upper band of the error

$$Error = \left| \frac{f^{n+1}(c)}{n+1!} \cdot (x-a)^{n+1} \right| = \frac{e^c}{5!}(x-a)^5 = \frac{e^c}{5!} \cdot -0'00792;$$

$$e^c \cdot -6'6 \cdot 10^{-5} \leq e^0 \cdot 6'6 \cdot 10^{-5} = 6'6 \cdot 10^{-5}$$

c) Table f

x_i	$f(x_i)$		
-1	$0'367879$	$A = 0'477304$	$C = 0'309634$
$-0'5$	$0'606531$	$B = 0'786938$	
0	1		

$$A = \frac{0'606531 - 0'367879}{-0'5+1} = 0'477304 \qquad C = \frac{0'786938 - 0'477304}{0+1} = 0'309634$$

$$B = \frac{1 - 0'606531}{0 - -0'5} = 0'786938$$

d) Estimate $e^{-0'38}$ by linear and quadratic

Linear $n = 1$; $n+1 = 2$ points $x_0 = -0'5 < x < x_1 = 0$

$P_1(x) = 0'606531 + 0'786938(x+0'5)$

$P_1(-0'38) = 0'606531 + 0'786938(0'12) = 0'700963$

Quadratic $n = 2$; $n+1 = 3$ points $x_0 = -1$; $x_1 = -0'5$; $x_2 = 0$

$-0'5 < c < 0$

$P_1(x) = 0'606531 + 0'786938(x+0'5) + 0'309634(x+0'5)(x-0)$

$P_1(-0'38) = 0'606531 + 0'786938(0'12) + 0'309634(0'12)(-0'38) = 0'66974\ldots$

$$Error_L = \left| \frac{f^{n+1}(c)}{n+1!} \cdot (x-x_0)(x-x_1) \right| = \frac{e^x}{2!}(0'12)(0'38) \leq \frac{e^0}{2!} 0'12 \, 0'38 = 0'0228 \simeq 2'28 \cdot 10^{-2}$$

$$Error_q = \left| \frac{f^{n+1}(c)}{n+1!} \cdot (x-x_0)(x-x_1)(x-x_2) \right| \frac{e^x}{2!}(0'12)(0'38)(0'62) \leq \frac{e^0}{2!} \cdot 0'12 \, 0'38 \, 0'62 = 0'014 \simeq 1'4 \cdot 1

Interpolation

$f(x) = \cos(x)$

a) Table

x_i	$f(x_i)$			
0	1			
		$A = -0'1244$		
0'25	0'9689		$C = -0'4816$	
		$B = -0'3652$		
0'5	0'8776			~~$E = -0'1436$~~
		~~$D = -0'5836$~~		
~~0'75~~	~~0'7317~~			

$$E = \frac{-0'5836 + 0'3652}{0'5} = -0'14368$$

$$A = \frac{0'9689 - 1}{0'25} = -0'1244$$

$$C = \frac{-0'3652 + 0'1244}{0'5} = -0'4816$$

$$B = \frac{0'8776 - 0'9689}{0'5 - 0'25} = -0'3652$$

$$D = \frac{0'7317 - 0'8776}{0'75 - 0'5} = -0'5836$$

b) Approximation of $\cos(0'41)$

$\cos(0'41) = 0'91712\ldots$

Linear $n = 1$; $n+1 = 2$ points $\qquad x_0 = 0'25 < x = 0'41 < x_1 = 0'5$

$$P_1(x) = 0'9689 - 0'3652(x - 0'25)$$

$$P_1(0'41) = 0'9689 - 0'3652(0'41 - 0'25) = 0'910468$$

Quadratic $n = 2$; $n+1 = 3$ points $\qquad x_0 = 0,\ x_1 = 0'25,\ x_2 = 0'5$

$$P_1(x) = 0'9689 - 0'3652(x - 0'25) - 0'4816 \cdot (x - 0'25)(x - 0'5)$$

$$P_1(0'41) = 0'9689 - 0'3652(0'41 - 0'25) - 0'4816(0'41 - 0'25)(0'41 - 0'5) = 0'917403\ldots$$

b) Error upper band (quadratic)

$$\text{Error} = \left| \frac{f^{(n+1)}(c)}{n+1!} \cdot (x - x_0)(x - x_1)(x - x_2) \right| = \left| \frac{f^3(c)}{3!} (x - 0)(x - 0'25)(x - 0'5) \right|$$

$$= \frac{\sin c}{6} \cdot 0'41 \cdot 0'16 \cdot 0'09 = \sin c \cdot 9'84 \cdot 10^{-4} < \sin 0'5 \cdot 9'84 \cdot 10^{-4}$$

$$= 4'71 \cdot 10^{-4}$$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$0 < c < 0'5$

sin x increasing

Interpolation

$f(x) = \sin(x)$

x_i	$f(x_i)$		
* 0'25	0'24740		
		A = 0'92812	
0'5	0'47943		C = -0'23856
		B = 0'80884	
0'75	0'68164		

$$A = \frac{0'47943 - 0'2474}{0'5 - 0'25} = 0'92812$$

$$C = \frac{0'80884 - 0'92812}{0'75 - 0'25} = -0'23856$$

$$B = \frac{0'68164 - 0'47943}{0'75 - 0'5} = 0'80884$$

b) Approximate $\sin 0'45$ → Linear (2 points) $x_0 = 0'25 < x < x_1 = 0'75$
→ quadratic (3 points) $x_0 = 0'25 < x_1 = 0'5 < x_2 = 0'75$

$\sin 0'45 = 0'43449\ldots$

Linear $n = 1$; $n+1 = 2$ points

$P_1(x) = 0'24740 + 0'92812\,(x - 0'25)$

$P_1(0'45) = 0'24740 + 0'97943\,(0'45 - 0'25) = 0'433024$

Quadratic $n = 2$; $n+2 = 3$ points

$P_1(x) = 0'24740 + 0'92812\,(x - 0'25) - 0'23856\,(x - 0'25)(x - 0'5)$

$P_1(0'45) = 0'2474 + 0'92812\,(0'45 - 0'25) - 0'23856\,(0'45 - 0'25)\cdot(0'45 - 0'5)$
$= 0'43540$

c) upper band error (quadratic)

$f'(x) = \cos x$
$f''(x) = -\sin x$
$f'''(x) = -\cos x$

$$\text{Error} = \left| \frac{f^{(n+1)}(c)}{n+1\,!} \cdot (x - x_0)(x - x_1)(x - x_2) \right|$$

$$\text{Error} = \left| \frac{f^3(c)}{3!} \cdot (0'45 - 0'25)(0'45 - 0'5)(0'45 - 0'75) \right| \quad \underline{0'25} < c < 0'75$$

cos c decrease

$$= \frac{\cos c}{6} \cdot 0'2 \cdot 0'05 \cdot 0'3 = \cos c \cdot 5 \cdot 10^{-4} < \cos 0'25 \cdot 10^{-4} \cdot 5$$

$= 4 \cdot 10^{-4}$

d) Approximate $\sin(-0'3)$; Linear and quadratic interpolation

Linear $n=1$; $n+1=2$ points $\qquad x_0=-0'5 < x < x_1=0$

$P_1(x): -0'47943 + 0'95886\,(x+0'5)$

$P_1(-0'3): -0'47943+0'95886\,(0'2): 0'2876\ldots$

Quadratic $n=2$; $n+1=3$ points $\qquad x_0=-1;\ x_1=-0'5,\ x_2=0$

$P_1(x) = -0'47943+0'95886\,(x+0'5)+0'23478\,(x+0'5)(x-0)$

$P_1(0'2): -0'47943+0'95886\,(0'2)+0'23478\,(0'2)(-0'3)=-0'30174\ldots$

e) bound errors

Linear $\qquad -0'5 < c < 0$

Upper bound?

$$\text{Error} = \left|\frac{f^{(n+1)}(c)}{n+1!}(x-x_0)(x-x_1)\right| = \left|\frac{f^{2}(c)}{2!}(-0'3+0'5)(-0'3-0)\right| = \frac{|\sin c|}{2!}\,0'2\cdot 0'3 \le$$

$$\frac{|\sin -0'5|}{2!}\cdot 0'2\cdot 0'3 = 0'0143\ldots \simeq 1'4\cdot 10^{-2}$$

Quadratic

$$\text{Error} = \left|\frac{f^{(n+1)}(c)}{n+1!}(x-x_0)(x-x_1)(x-x_2)\right| = \frac{\cos c}{3!}(-0'3+1)(-0'3+0'5)(-0'3-0) =$$

$$= \frac{\cos c}{3!}(0'7)(0'2)(0'3) \le \frac{0'7\cdot 0'2\cdot 0'3}{3!} = 0'007 \simeq 7\cdot 10^{-3}$$

Equations

2/ $e^{-3x}-9x=0 \qquad f(x)=e^{-3x}-9x$ {cont // diff

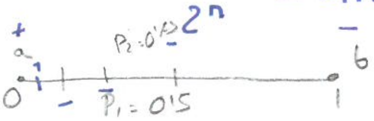
a) Only one root $[a, a+1]$

$f'(x) = -3e^{-3x}-9 < 0 \qquad f(0)=1>0 \qquad [0,1]$ there's one root

strictly decreasing on $\mathbb{R}$ $\qquad f(1) = -8'95 < 0$

b) Bisection method, approx of p with error less than 0'1

$\frac{b-a}{2^n} <$ Error; $\frac{1-0}{2^n} < 0'1$; $10 < 2^n$; $\log 10 < n \log 2$; $\frac{\log 10}{\log 2} < n$



$0'125/2 \qquad \frac{0'25+0}{2}$

$3'32 < n$; $n=4$

$P_1=0'5 \quad f(0'5): -$; $P_2=0'25 \quad f(0'25)= -$; $P_3=0'125 \quad f(0'125)=-$

$P_4=0'0625 \quad f(0'0625)=+ \qquad [0'0625,\ 0'125]$

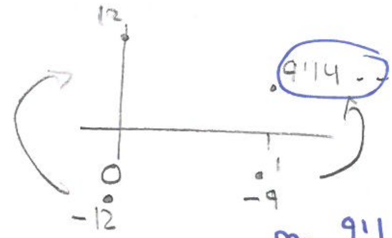
c) Newton's method → Conditions Up to 10^{-5}

1) $f(0) \cdot f(1) < 0$

2) Sign of $f'(x) = -3e^{-3x} - 9 < 0 \quad \forall x \in [0,1]$

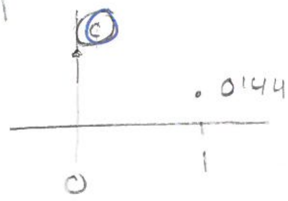
3) $f''(x) = 9e^{-3x} > 0$

m



m= 9'1

M



M = 9

P(0) choice

$F(p_0) > 0 \quad f(0) = 1$ = Same sign $\quad p_0 = 0$

$f''(p_0) > 0 \quad f''(0) = 9$

$$P_1 = P_0 - \frac{f(p_0)}{f'(p_0)} = 0 - \frac{1}{-12} = 0'08333$$

$$P_2 = P_1 - \frac{f(p_1)}{f'(p_0)} = 0'08333 - \frac{0'0288}{-11'3364} = 0'085873...$$

$$P_3 = P_2 - \frac{f(p_2)}{f'(p_2)} = 0'085873 - \frac{0'0000326}{-11'3186} = 0'085875...$$

$$|P - P_1| \leq \frac{M}{2m} |P_n - P_{n-1}|^2$$

$$|P - P_2| \leq \frac{9}{2 \cdot 9'1} |P_2 - P_1|^2 \Rightarrow \frac{9}{2 \cdot 9'1} \cdot |0'085873 - 0'08333|^2;$$

$$|P - P_1| \leq \frac{9}{2'91} \cdot 6'451 \cdot 10^{-6} = 3'19 \cdot 10^{-6}$$

$$|P - P_1| \leq \frac{m}{2m} \cdot |P_1 - P_0| = \frac{9}{2 \cdot 9'1} \cdot |0'08333 - 0|^2 = 3'4\,10^{-3}$$ (more than needed)

c) Linear upper band error

$$-0'6 < c < -0'4$$

$$Error = \left| \frac{f^{(n+1)}(c)}{(n+1)!} \cdot (x-x_0)(x-x_1) \right| = \left| \frac{f^2(c)}{2!} (-0'45+0'6)(-0'45+0'4) \right|$$

$$= \frac{e^c}{2!} \cdot 0'15 \cdot 0'05 \leq \frac{e^{-0'4}}{2!} \cdot 0'15 \cdot 0'05 = 0'0025 = 2'5 \cdot 10^{-3}$$

d) Quadratic upper band error

$$Error = \left| \frac{f^{(n+1)} c}{(n+1)!} \cdot (x-x_0)(x-x_1)(x-x_2) \right| = \left| \frac{f^3(c)}{3!} (-0'45+0'6)(-0'45+0'4)(-0'45+0'2) \right|$$

$$= \frac{e^c}{3!} 0'15 \cdot 0'05 \cdot 0'25 \leq \frac{e^{-0'4}}{3!} \cdot 0'15 \cdot 0'05 \cdot 0'25 = 0'00029 = 2'1 \cdot 10^{-4}$$

Taylor and interpolation

1/ $f(x) = \sin x$

a) 4-order - Maclaurin to approximate $\sin(-0'3)$

$$P(x) = \cancel{\sin x} + \cos(x)(x-a) - \frac{\cancel{\sin(x)}}{2!}(x-a)^2 - \frac{\cos(x)}{3!}(x-a)^3 + \frac{\cancel{\sin(x)}}{4!}(x-a)^4$$

$$P_4(x) = 1 \cdot (x-0) - \frac{x^3}{3!} = x - \frac{x^3}{3!}$$

$$P_4(-0'3) = -0'3 - \frac{(-0'3)^3}{3!} = -0'2955$$

b) Upper band on the error

$$-0'3 < c < 0$$

$$Error_4 = \left| \frac{f^{4+1}(c)}{4+1!} \cdot (x-a)^5 \right| = \frac{\cos c}{5!} (0'3)^5 \leq \frac{(0'3)^5}{5!} = 2'02 \cdot 10^{-5}$$

0

c) Table of the divided differences

x_i	$f(x_i)$		
-1	$-0'84147$		
		A: 0'72405	
$-0'5$	$-0'47943$		C: 0'23478
		B: 0'45886	
0	0		

$$A = \frac{-0'47943 + 0'84147}{-0'5+1} = 0'72408$$

$$B = \frac{0 + 0'47943}{0+0'5} = 0'45886$$

$$C = \frac{0'45886 - 0'72408}{0+1} = 0'23478$$

Interpolation and Taylor

$f(x)=e^x$

1/ approximation of $e^{-0'45}$ 3 order Maclaurin and upper bound error $a=0$ $x=-0'45$

$$P(a)=f(a)+f'(a)\cdot(x-a)+\dots\ \frac{f^n(a)}{n!}(x-a)^n$$

$$P_3(x)=e^x+e^x(x-a)+\frac{e^x}{2!}(x-a)^2+\frac{e^x}{3!}(x-a)^3=$$

$$=e^0+e^0x+\frac{e^0x^2}{2!}+\frac{e^0x^3}{3!}=x+\frac{x^2}{2!}+\frac{x^3}{3!}$$

$$P_3(-0'45)=-0'45+\frac{-0'45}{2!}+\frac{-0'45}{3!}=0'63606$$

$-0'45<c<0$

$$Error_3=\left|\frac{f^4(c)}{4!}(x-a)^4\right|=\frac{e^c}{4!}\cdot 0'45^4\leq\frac{e^0}{4!}\cdot 0'45^4=1'7\cdot10^{-3}$$

a) Table $e(-0'45)$ $x=-0'45$

x_i	$f(x_i)$		
$-0'6$	$0'54881$		
		$A=0'60755$	
$-0'4$	$0'67032$		$C=0'33625$
		$B=0'74205$	
$-0'2$	$0'81873$		

$$A=\frac{0'67032-0'54881}{-0'4+0'6}=0'60755$$

$$C=\frac{0'74205-0'60755}{-0'2+0'6}=0'33625$$

$$B=\frac{0'81873-0'67032}{-0'2+0'4}=0'74202$$

Linear interpolating $n=1;\ n+1=2$ $x_0=-0'6<x<x_1=-0'4$

$P_1(x)=0'54881+0'67032\cdot(x+0'6)$

$P_1(-0'45)=0'54881+0'67032(0'15)=0'644358$

Quadratic interpolating $n=2;\ n+1=3$

$P_1(x)=0'54881+0'67032\cdot(x+0'6)+0'33625(x+0'2)(x+0'6)$

$P_1(-0'45)=0'54881+0'67032\cdot(0'15)+0'33625(-0'25)(0'15)=0'636748625$

Taylor

$f(x) = \log(1+x)$

a) Maclaurin polynomial of f order 5

$$P_5(a) = \log(1+a) + (1+a)^{-1}(x-a) - \frac{(1+a)^{-2}}{2!}(x-a)^2 + \frac{2(1+a)^{-3}}{3!}(x-a)^3 - \frac{6(1+a)^{-4}}{4!}(x-a)^4 + \frac{24(1+a)^{-5}}{5!}(x-a)^5 =$$

$$= 0 + x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} + \frac{24x^5}{5!}$$

b) approx of $\log(1'2)$; upper band $\log(1+x) = \log(1+0'2)$

$$P_5(1'2) = 0'2 - \frac{0'2^2}{2!} + \frac{2\cdot(0'2)^3}{3!} - \frac{6\cdot(0'2)^4}{4!} + \frac{24(0'2)^5}{5!} = 0'182330 \qquad 0 < c < 0'2$$

$$\text{Errors} = \left|\frac{f^6(c)}{6!}\cdot(x-a)^6\right| = \left|\frac{120(c)^{-6}}{6!}\cdot(0'2-0)^6\right| \leq \frac{120\cdot 0'2^6}{6!} = 1'07\cdot 10^{-5}$$

c) Approx $\log(0'5)$ $\log(1+x)$; $x=-0'5$

$$P_5(x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} \qquad -0'5 < c < 0$$

$$P_5(x) = -0'5 - \frac{(-0'5)^2}{2!} + \frac{2\cdot -0'5^3}{3!} - \frac{6(-0'5)^4}{4!} + \frac{24(-0'5)^5}{5!} = -0'688541\ldots$$

$$\text{Errors} = \left|\frac{f^6(c)}{6!}\cdot(x-a)^6\right| = \left|\frac{5!(1+c)^{-6}}{6!}\cdot(-0'5-a)^6\right| \leq \frac{5!\cdot(-0'5)^{-6}}{6!}\cdot(-0'5)^6 = 0'16

Taylor

$f(x) = \sin(x)$

a) approx to $\sin(-0'4)$ (rads) Maclaurn of orders 5 and 6 $a = 0$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^n(a)}{n!}(x-a)^n$$

$$P_6(x) = \sin a + \cos(a)(x-a) - \frac{\sin(a)}{2!}(x-a)^2 - \frac{\cos(a)}{3!}(x-a)^3 + \frac{\sin(a)}{4!}(x-a)^4 + \frac{\cos(a)}{5!}(x-$$

$$- \frac{\sin(a)}{6!}(x-a)^6$$

1- Maclaurin approx to $x = -0'4$ $a = 0$ 5 and 6 → the same

$$P_5 = \cancel{\sin(0)}^{0} + \cos 0 (x-0) - \frac{\cancel{\sin 0}^{0}}{2!}(x-0)^2 - \frac{\cos 0}{3!}(x-0)^3 + \frac{\cancel{\sin 0}^{0}}{4!}(x-0)^4 + \frac{\cos 0}{5!}(x-0)^5$$

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} = -0'4 - \frac{(-0'4)^3}{6} + \frac{(-0'4)^5}{5!} = -0'38941666$$

b) Error for $n = 5$ and $n = 6$

$$\text{Error}_n = \left| \frac{f^{n+1}(c)}{n+1!} \cdot (x-a)^{n+1} \right|$$

$$\text{Error}_5 = \left| \frac{f^6(c)}{6!} \cdot (x-a)^6 \right| \leq \frac{1}{6!}(x-a)^6 \Rightarrow \left| \frac{1}{6!}(-0'4-0)^6 \right| = 5'688 \cdot 10^{-6}$$

$$\text{Error } 6 = \left| \frac{f^7(c)}{7!}(x-a)^7 \right| \leq \frac{1}{7!}(x-a)^7 \Rightarrow \left| \frac{1}{7!}(-0'4-0)^7 \right| = 3'250 \cdot 10^{-7}$$

c)

Taylor

$f(x) = \cos x$

a) Maclaurin polinomial of order 5 ($n=5$) $a=0$

$$P_5(x) = \cos(a) - \cancel{\sin(a)}(x-a) - \frac{\cos a}{2!}(x-a)^2 + \cancel{\frac{\sin a}{3!}}(x-a)^3 + \frac{\cos a}{4!}(x-a)^4 - \cancel{\frac{\sin a}{5!}}(x-a)^5$$

$$P_5(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

b) Approximation of $\cos(-0'3)$ for the 5-order

$$P_5(-0'3) = 1 - \frac{(-0'3)^2}{2!} + \frac{(-0'3)^4}{4!} = 0'9553375...$$

$-0'3 < c < 0$

$$Error_5 = \left|\frac{f^6(c)}{6!}(x-a)^6\right| = \left|\frac{-\cos(c)}{6!}(x-a)^6\right| \leq \frac{1}{6!}(-0'3-0)^6$$

$= 1 \cdot 10^{-6}$

c) $\cos 25^\circ$ estimate for 4-order maclaurin. Upper bound of the error

$$25^\circ = 25 \cdot \frac{2\pi}{360} = 0'43 = x$$

$P_4(x) = P_5(x)$

$$P_4(0'43) = 1 - \frac{(0'43)^2}{2!} + \frac{(0'43)^4}{4!} = 0'90631734$$

$$Error_4 = \left|\frac{f^5(c)}{5!} \cdot (x-a)^5\right| = \left|\frac{\sin c}{5!}(0'43-0)^5\right| \leq \frac{1}{5!}(0'43)^5 = 1'22 \cdot 10^{-4}$$

$\frac{0}{2} = 0$

$\frac{3}{5} < 0$

-1

$\frac{2n+3}{n+1}$

$\frac{4}{6}$

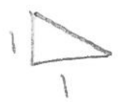
$\frac{2(n+1)-(2n+3)}{(n+1)^2}$

$0^2 - 1$

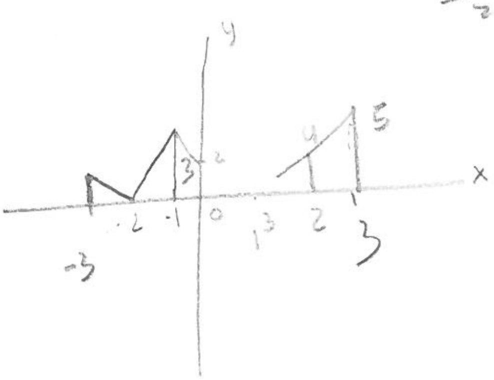
2

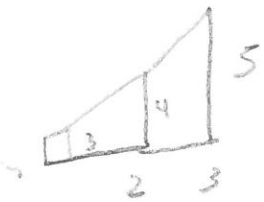
$\frac{1 \cdot 1}{2} \to 1/2$

$-7 \cdot e^{-7}$



$\frac{1 \cdot 3}{2} = 3/2$





$f(x) = \frac{\cos x}{x}$ $\quad I = \int_1^2 \frac{\cos x}{x} dx$ $\quad |f^4(x)| \leq 24 \ \forall [1,2]$

a) Upper bound of $|f''(x)| \ \forall x \in [1,2]$

$f' = \frac{-\text{sen}x \cdot x - \cos x}{x^2}$; $f'' = \frac{(-x\cos x - \cancel{\text{sen}x} + \cancel{\text{sen}x})x^2 - (-\text{sen}x \cdot x - \cos x)2x}{x^4} =$

$= \left| \frac{-x^2\cos x + 2x\,\text{sen}x + 2\cos x}{x^3} \right| =$

$f''|(1)| = 0'062\ldots$

$f''|(2)| = 0'53\ldots$ $\quad = \frac{|-x^2\cos x|}{x^3} + \frac{|2x\,\text{sen}x|}{x^3} + \frac{|2\cos x|}{x^3}$

$\pi = \int_0^1 \frac{4}{1+x^2} dx$ up to an error of 10^{-2}

$|I - I^*| \leq \frac{M_2(b-a)^3}{12 \cdot n^2} < tol$

$f'(x) = \frac{-8x}{(1+x^2)^2} \rightarrow 1 + x^2 + 2x \rightarrow 2x + 2$

$f''(x) = \frac{-8(1+x)^2 \cdot}{(1+x^2)^4}$

$= \frac{}{(1+x^2)^4} =$

$= \frac{}{(1+x^2)^4}$

1/ Approx of $\log(2) = \int_1^2 \frac{1}{x} dx$ up to an error of $5 \cdot 10^{-7}$

a) Trappezoid rule

$\text{Error} \leq \frac{M_2(b-a)^3}{12n^2} < 5 \cdot 10^{-7}$

$f'(x) = -\frac{1}{x^2}$; $f' = -x^{-2}$; $f'' = -2x^{-3} = \frac{2}{x^3}$ $\xrightarrow{x=1} 2$; $f''' = 2x^{-3} = -6x^{-4}$

$f^{IV} = +24x^{-5}$

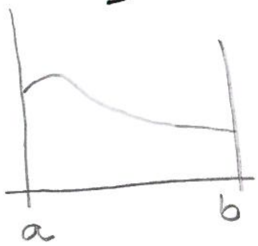
$\frac{2 \cdot (2-1)}{12n^2} < 5 \cdot 10^{-7}$; $\frac{2}{5 \cdot 10^{-7}} < 12n$; $577'35 < n$; $n = 578$ times

b) Simpsone rule

$\text{Error} \leq \frac{M_4(b-a)^5}{90n^4 \ 2^5} < 5 \cdot 10^{-7}$; $\frac{24(2-1)^5}{90n^4 2^5} < 5 \cdot 10^{-7}$; $\frac{\cdots}{90 \cdot 2^5} < n^4$; $11'36 < n = 12$

Integral calculus

→ Definite integral (area)

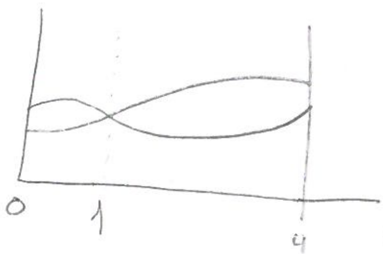


$[a,b]$ $\qquad \int_a^a f(x)\,dx = 0$, then, $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

$$\int_a^b f(x) = F(b) - F(a) = [F(x]_a^b$$

1/ $y = x^2$; $y = \sqrt{x}$ between $x = 0$ & $x = 4$

$x^2 - \sqrt{x} = 0$; $x = 1$



	Subintervals	Points to evaluate
Trapezoid	n	$n+1$
Simpsone	$2n$	$2n+1$

Trapezoid method

$\int_2^3 (4x^2+6)\,dx$ $\qquad n = 4$ $\qquad h = \frac{b-a}{n} = \frac{3-2}{4} = 1/4$

$$\approx \frac{h}{2}\left[\underset{2}{F(x_0)} + 2\underset{2+1/4}{f(x_1)} + 2\underset{2+1/4+1/4}{f(x_2)} + 3\underset{2+1/4+1/4+1/4}{f(x_3)} + \underset{3}{f(x_4)}\right]$$

$$\approx \frac{1}{8}\left[22 + \frac{105}{2} + 62 + \frac{145}{2} + 42\right] = 31'375$$

$\int_1^2 \frac{1}{x}\,dx$ $\qquad n = 10$ $\qquad \Delta x = \frac{b-a}{n} = \frac{2-1}{10} = 1/10$

$$\int_a^b \frac{\Delta x}{2}\left[f(x_0) + 2\sum^{n=10}(F(x_i) + f(x_n)\right] = \frac{1}{20}\left[f_1(1) + 2f_2(1'1) + 2f_3(1'2) + \dots + f(2)\right] = 0'693$$

Simpsone method

$\Delta x = \frac{b-a}{n}$ $\quad n = 4$ $\quad S_n = \frac{\Delta x}{3}\left[\underset{2}{f(x_0)} + 4\underset{4}{f(x_1)} + 2\underset{6}{f(x_2)} + 4\underset{8}{f(x_3)} + 2f(x_4) + \dots + 4f(x_{n-1}) + \underset{10}{f(x_n)}\right]$

$\int_2^{10} x^3\,dx$ $\quad \Delta x = \frac{10-2}{4} = 2 = \frac{2}{3}\left[f(2) + 4f(4) + 2f(6) + 4f(8) + f(10)\right] \approx 2496$

1/ $\int \frac{1}{x^2 \cdot \sqrt[5]{x^2}} = \int x^{-2} \cdot x^{-2/5} = \int x^{-12/5} = \frac{x^{-7/5}}{-7/5} = 5/{-7 \cdot \sqrt[5]{x^7}}$

2/ $\int (x+2)^3 dx = \int u^3 = \frac{u^4}{4} = \frac{(x+2)^4}{4}$

$u = (x+2)$

$dx = -$

3/ $\int (2x+1)(x^2+x+1) = \cancel{\int (2x^3+2x^2+2x+x^2+x+1 = \int 2x^3+3x^2+3x+1} =$

4/ $\int x \cdot \sin x \, dx$ ALPES $= x \cdot \cos x - \int -\cos x = -x \cdot \cos + \operatorname{sen} x + C$

$u = x \quad du = 1$

$dv = \sin x \quad v = -\cos x$

$+\frac{1}{2} \int \frac{x^{-2}}{x} = +\frac{1}{2} \int x^{-3}$

5/ $\int \frac{\ln x}{x^3} dx$ ALPES $= \frac{-\ln x \cdot x^{-2}}{2} - \int \frac{-x^{-2}}{2x} = \frac{-\ln x \cdot x^{-2}}{2} + \frac{x^{-2}}{-4} = -\frac{\ln x}{2x} - \frac{1}{4x^2}$

$u = \ln x \quad du = \frac{1}{x}$

$dv = x^{-3} \quad v = -\frac{x^{-2}}{2}$

6/ $\int (x^3+5x^2-2)e^{2x} dx$ ALPES $= \frac{(x^3+5x^2-2)e^{2x}}{2} - \int \frac{e^{2x}}{2} \cdot 3x^2+10x$

$u = x^3+5x^2-2 \quad du = 3x^2+10x$

$dv = e^{2x} \quad v = \frac{e^{2x}}{2}$

$u = 3x^2+10x \quad du = 6x+10$

$dv = \frac{e^{2x}}{2} \quad v = \frac{e^{2x}}{4}$

$= \frac{(x^3+5x^2-2)e^{2x}}{2} - \left[\frac{3x^2+10x \cdot e^{2x}}{4} - \int \frac{e^{2x}}{4} \cdot 6x+10 \right.$

7/ $\int \ln x \, dx$ ALPES $= x \cdot \ln x - \int x \cdot \frac{1}{x} = x \cdot \ln x - x + C$

$u = \ln x \quad du = \frac{1}{x}$

$dv = 1 \quad v = x$

8/ $\int e^x \cos x \, dx$ ALPES $= e^x \operatorname{sen} x - \int e^x \operatorname{sen} x = e^x \operatorname{sen} x + e^x \cos x + \int -\cos x \, e^x =$ -1

$u = e^x \quad du = e^x$

$dv = \cos x \quad v = \operatorname{sen} x$

$u = e^x \quad du = e^x$

$dv = \operatorname{sen} x \quad v = -\cos x$

$$\int e^x \cos x \, dx = e^x \operatorname{sen} x + e^x \cos x - \int e^x \cos x \, dx; \; 2\int e^x \cos x \, dx = e^x \operatorname{sen} x + e^x \cos x;$$

$$\int e^x \cos x = \frac{e^x}{2}(\operatorname{sen} x + \cos x) + C$$

9/ $\int x\sqrt{1+x} \, dx = \int (u-1) \cdot \sqrt{u} = \int u\sqrt{u} \, du - \int \sqrt{u} \, du = \int u^{3/2} du - \int u^{1/2} du = \frac{u^{5/2}}{\frac{5}{2}} - \frac{u^{3/2}}{\frac{3}{2}}$

$u = 1+x; \quad dx = du$

$x = u-1$

$$= \frac{2(1+x)^{5/2}}{5} - \frac{2(1+x)^{3/2}}{3} + C$$

10) $$\int \frac{1}{5+5x^2} = \int \frac{1/5}{5/5 + \frac{5x^2}{5}} = \frac{1}{5}\int \frac{1}{1+x^2} = \frac{1}{5}\cdot \mathrm{Arctg}\, x + C$$

11) $$\int \frac{1}{1+16x^2}\,dx = \frac{1}{4}\cdot\int \frac{4}{1+(4x)^2} = \frac{1}{4}\cdot \mathrm{Arctg}\, 4x + C$$

12) $$\int \frac{\cos x}{1+\mathrm{sen}^2 x} = \mathrm{Arctg}\,\mathrm{sen}\, x + C$$

13) $$\int \frac{x^2}{1+x^6}\,dx = \int \frac{x^2}{1+(x^3)^2} = \frac{1}{3}\int \frac{3x^2}{1+(x^3)^2} = \frac{1}{3}\cdot \mathrm{Arctg}\, x^3 + C$$

14) $$\int \frac{3}{1+9x^2}\,dx = \mathrm{Arctg}\, 3x + C$$

15) $$\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{(x+1/2)^2+3/4} = \frac{2}{\sqrt{3}}\,\mathrm{Arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

16) $$\int \mathrm{tg}\, x\, dx = \int \frac{-\mathrm{sen}\, x}{\cos x} = -\ln \cos x + C$$

17) $$\int \frac{1}{x \ln x}\,dx = \int \frac{1/x}{\ln x} = \ln(\ln x) + C$$

18) $$\int \frac{x+1}{x-5}\,dx = \int \frac{x+1-5+5}{x-5} = \int \frac{x-5}{x-5}\,dx + \int \frac{6}{x-5} = x + 6\ln(x-5) + C$$

Integrales arcotg

$$\int \frac{f'}{1+f^2} = \text{arctg } f \qquad x^2+6x+9=(x+3)^2$$

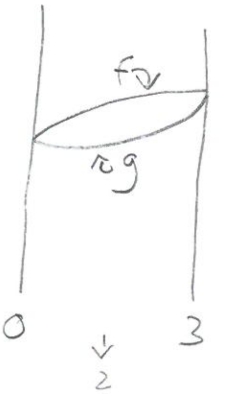
a) $\int \frac{1}{85+48x+8x^2}\,dx = \int \frac{1/8}{85/8+48x/8+8x^2/8} = \int \frac{1/8}{85/8+6x+x^2} = \int \frac{1/8}{\frac{85}{8}+6x+x^2+9-9}\,dx =$

$= \int \frac{1/8}{\underbrace{\frac{85}{8}-9}_{13/8}+(x+3)^2} = \int \frac{1/8\cdot 8\cdot \frac{1}{13}}{\frac{1}{13}\cdot\frac{13}{8}\cdot 8+\frac{8(x+3)^2}{13}} = \int\frac{1/13}{1+\frac{8(x+3)^2}{13}} = \int \frac{1/13}{1+\left[\frac{\sqrt{8}(x+3)}{\sqrt{13}}\right]^2} = \frac{1}{13}\cdot\sqrt{\frac{13}{8}}\cdot\int\frac{\sqrt{8}/\sqrt{13}}{1+\left[\frac{\sqrt{8}(x+3)}{\sqrt{13}}\right]^2}$

$= \frac{\sqrt{13}}{13\sqrt{8}}\,\text{arctg}\,\frac{\sqrt{8}}{\sqrt{13}}(x+3)+C$

1) $f(x)=3+2x-x^2 \qquad g(x)=x^2-4x+3$

$3+2x-x^2 = x^2-4x+3;\ -2x^2+6x=0;\ \overset{0}{x}(\overset{3}{-2x+6})=0$



$f(2)=3$
$g(2)=-1$

$A=\int_0^3 (3+2x-x^2)-(x^2-4x+3)\,dx;\ A=\int_0^3 -2x^2+6x\,dx =$

$= \left[\frac{-2x^3}{3}+\frac{6x^2}{2}\right]_0^3 = \frac{-2\cdot 3^2}{3}+\frac{6\cdot 3^2}{2}-0+0 = 9u^2$

a) $\int \frac{dx}{1+4x^2} = \frac{1}{2}\int \frac{1\cdot 2}{1+(2x)^2} = \frac{1}{2}\text{Arctg } 2x + C$

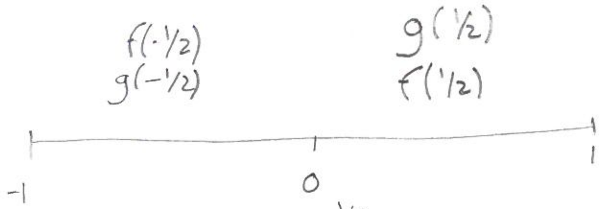
b) $\int \frac{dx}{1+5x^2} = \int \frac{1}{1+(\sqrt{5}x)^2} = \frac{1}{\sqrt{5}}\int \frac{\sqrt{5}\cdot 1}{1+(\sqrt{5}x)^2} = \frac{1}{\sqrt{5}}\cdot \text{Arctg } \sqrt{5}x$

c) $\int \frac{dx}{9+4x^2} = \int \frac{1/9}{9/9+\frac{4x^2}{9}} = \int \frac{1/9}{1+\left(\frac{2x}{3}\right)^2} = \frac{1}{9}\int \frac{1}{1+\left(\frac{2x}{3}\right)^2} = \frac{1}{9}\cdot\frac{3}{2}\int\frac{2/3}{1+\left(\frac{2x}{3}\right)^2} = \frac{1}{6}\cdot \text{Arctg}\left(\frac{2x}{3}\right)$

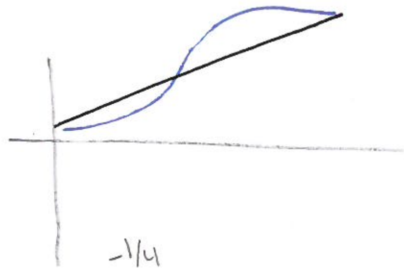
d) $\int \frac{dx}{3x^2+11} = \int \frac{1/11}{\frac{3x^2}{11}+1} = \frac{1}{11}\int \frac{1}{\left(\frac{\sqrt{3}x}{\sqrt{11}}\right)^2+1} = \frac{1}{11}\cdot\frac{\sqrt{11}}{\sqrt{3}}\int \frac{\sqrt{3}/\sqrt{11}}{\left(\frac{\sqrt{3}x}{\sqrt{11}}\right)^2+1} = \frac{\sqrt{33}}{33}\cdot \text{Arctg } \frac{\sqrt{3}x}{\sqrt{11}}$

1/ Area between $f(x) = x$ and $g(x) = \sqrt[3]{x}$

$x = \sqrt[3]{x}$; $x^3 = x$; $x^3 - x = 0$; $x(x^2-1) = 0$; $x=-1$, $x=0$, $x=1$



$\Rightarrow$

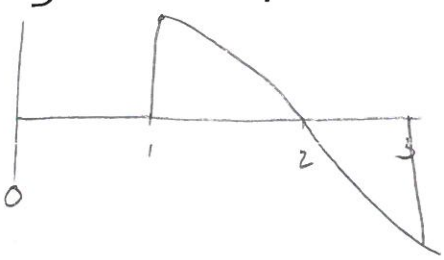


$x^{1/3}$

$$A = \int_{-1}^{0} x - \sqrt[3]{x} + \int_{0}^{1} \sqrt[3]{x} - x = \left[\frac{x^2}{2} - \frac{x^{4/3}}{4}\right]_{-1}^{0} + \left[\frac{x^{4/3}}{4} - \frac{x^2}{2}\right]_{0}^{1}$$

$x=0$; $x=2$; $x=4$

2/ $y = x(x-2)(x-4)$ $\to x^2 - 2x(x-4) = x^3 - 4x^2$ between $x=1$ and $x=3$



$$A = \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_1^2 - \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_2^3$$

$$= 7/2$$

$(x^2 - 2x)(x-4)$

$\int x^3 - 4x^2 - 2x^2 - 8x$

$\frac{x^4}{4} - \frac{6x^3}{3} - \frac{8x^2}{2} = \frac{x^4}{4} - 2x^3 + 4x^2$

$0'94 \cdot 0'3 + 2'75 \cdot 0'35 + x \cdot 0'35 = 4$

$1'24 + 0'35x = 4$

$0'35x = 2'7555$

$x = 7'8728\ldots$

$3'44 \cdot 0'3 + 5'75 \cdot 0'35 + x \cdot 0'35 = 4$

$3'0445 + 0'35x = 4$

$0'35x = 0'9555$

$x = 2'73$

Integrales logarítmicas

$\int \frac{1}{x} dx = \ln|x| + C$

a) $\int \frac{3}{x} dx = 3\int \frac{1}{x} dx = 3\ln|x| + C$

b) $\int \frac{x}{x^2+1} dx = \frac{1}{2}\int \frac{2x}{x^2+1} = \frac{1}{2} \cdot \ln|x^2+1| + C$

c) $\int tg x \, dx = -\int \frac{-\operatorname{sen} x}{\cos x} = -\ln|\cos x| + C$

d) $\int \frac{x-2}{x^2-4x} dx = \frac{1}{2}\int \frac{2(x-2)}{x^2-4x} dx = \frac{1}{2} \cdot \ln|x^2-4x| + C$

e) $\int \frac{e^{2x}}{1-e^{2x}} dx = -\frac{1}{2}\int \frac{-2e^{2x}}{1-e^{2x}} = \frac{1}{2}\ln|1-e^{2}$

f) $-\int \frac{-\operatorname{sen} x - \cos x}{\operatorname{sen} x + \cos x} dx = -\ln|\operatorname{sen} x + \cos x| + C$

g) $\int \frac{1}{x \ln x} = \int \frac{\frac{1}{x}}{\ln x} x = \ln|\ln x| + C$

h) $\int \frac{1}{\sqrt{x} \cdot (1+\sqrt{x})} = 2\ln|1+\sqrt{x}| + C$

S_e

AB \ CD	00	01	11	10
00	1			1
01				1
11	1	1	1	1
10	1		1	1

$= \bar{D}C + \bar{D}\bar{B} + CA + BA$

S_f

AB \ CD	00	01	11	10
00	1			
01	1	1		1
11	1		1	1
10	1	1	1	1

$= \bar{D}\bar{C} + \bar{C}B\bar{A} + \bar{B}A + \bar{D}CB + CA$

S_g

AB \ CD	00	01	11	10
00			1	1
01	1	1		1
11		1	1	1
10	1	1	1	1

$= \bar{B}A + \bar{D}C + C\bar{B}\bar{A} + \bar{C}B\bar{A} + DA$

$$\int \frac{2x+7}{x^2-5x+11} = \int \frac{2x-5+5+7}{x^2-5x+11} = \int \frac{2x-5}{x^2-5x+11} + \int \frac{12}{x^2-5x+11}$$

$\to Ln$ $\quad \to A.ctg$

$x^2 - 5x + \frac{25}{4} = \left(x-\frac{5}{2}\right)^2$

$a^2 + b = 11$, $b = \frac{19}{4}$

$$\int \frac{2x-5}{x^2-5x+11} = Ln\,|x^2-5x+11| + C$$

$$\int \frac{12}{x^2-5x+11} = \int \frac{12}{\left(x-\frac{5}{2}\right)^2+19/4} = 12\int \frac{1}{\left(x-\frac{5}{2}\right)^2+19/4} = 12\int \frac{4/19}{1+\frac{4}{19}\cdot\left(x-\frac{5}{2}\right)^2} = \frac{48}{19}\int \frac{1}{1+\left[\frac{2}{\sqrt{19}}\cdot\left(x-\frac{5}{2}\right)\right]^2}$$

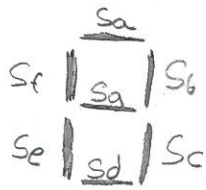
$$= \frac{48}{19}\int \frac{1}{1+\left(\frac{2x-5}{\sqrt{19}}\right)^2} = \frac{48}{19}\cdot\frac{\sqrt{19}}{2}\cdot\int \frac{2/\sqrt{19}}{1+\left(\frac{2x-5}{\sqrt{19}}\right)^2} = \frac{24\sqrt{19}}{19}\cdot Arctg\left(\frac{2x-5}{\sqrt{19}}\right)$$

$$Ln\,|x^2+5x+11| + \frac{24\sqrt{19}}{19}\cdot Arctg\left(\frac{2x-5}{\sqrt{19}}\right) + C$$

7

3/ Tabla de verdad conversor hexadecimal

A	B	C	D	Sa	Sb	Sc	Sd	Se	Sg	Sg	Output
0	0	0	0	1	1	1	1	1	1	0	0
0	0	0	1	0	1	1	0	0	0	0	1
0	0	1	0	1	1	0	1	1	0	1	2
0	0	1	1	1	1	1	1	0	0	1	3
0	1	0	0	0	1	1	0	0	1	1	4
0	1	0	1	1	0	1	1	0	1	1	5
0	1	1	0	1	0	1	1	1	1	1	6
0	1	1	1	1	1	1	0	0	0	0	7
1	0	0	0	1	1	1	1	1	1	1	8
1	0	0	1	1	1	1	1	0	1	1	9
1	0	1	0	1	1	1	0	1	1	1	A
1	0	1	1	0	0	1	1	1	1	1	B
1	1	0	0	1	0	0	1	1	1	0	C
1	1	0	1	0	1	1	1	1	0	1	D
1	1	1	0	1	0	0	1	1	1	1	E
1	1	1	1	1	0	0	0	1	1	1	F



Karnaugh

Sa

AB \ CD	00	01	11	10
00	1	1	1	1
01		1	1	1
11	1		1	1
10	1	1		1

$= \bar B\bar A + \bar C\bar B A + \bar D A + C\bar A + CB + D\bar A$

Sb

AB \ CD	00	01	11	10
00	1	1	1	1
01	1		1	
11		1		
10	1	1		1

$= \bar D\bar B + D\bar C A + \bar D\bar C\bar A + \bar B\bar A + DC\bar A$

Sc

AB \ CD	00	01	11	10
00	1		1	
01	1	1	1	1
11		1		
10	1	1	1	1

$= \bar B A + \bar D\bar C\bar A + B\bar A + DC\bar A$

Sd

AB \ CD	00	01	11	10
00	1		1	1
01		1		1
11	1	1		1
10	1	1	1	

$= \bar D\bar B\bar A + C\bar B\bar A + D\bar C B$

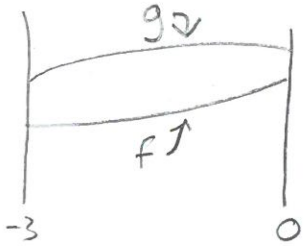
1/ $f(x) = \frac{x+3}{x^2+9}$ $\quad g(x) = \frac{1}{9}x + \frac{1}{3}$ $\quad (x+3)(x^2+9) = x^3 + 9x + 3x^2 + 27$

$x^3 + 3x^2 + 9x + 27$

Intersection points

$\frac{x+3}{x^2+9} = \frac{x}{9} + \frac{1}{3} \Leftrightarrow \frac{x+3}{x^2+9} = \frac{x+3}{9} \Longleftrightarrow 9\cancel{x} + \cancel{27} = x^3 + 3x^2 + 9\cancel{x} + \cancel{27} \Leftrightarrow x^3 + 3x^2 = 0 \Leftrightarrow$

$x^2(x+3) \Rightarrow \begin{array}{l} x=0 \\ x=-3 \end{array}$ $\quad [-3, 0]$



$f(-1) = \frac{1}{5}$

$g(-1) = \frac{2}{9}$

$A = \int_{-3}^{0} g(x) - f(x) = \int_{-3}^{0} \frac{x}{9} + \frac{1}{3} - \left(\frac{x+3}{x^2+9}\right) =$

$= \int \frac{\frac{x^2}{2}}{9} + \frac{x}{3} = \frac{x^2}{18} + \frac{x}{3}$

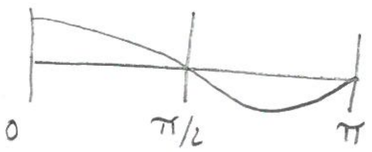
$\int \frac{x+3}{x^2+9} = \frac{1}{2}\int \frac{2x}{x^2+9} + \int \frac{3}{x^2+9} = \frac{1}{2} \ln|x^2+9| + \int \frac{3}{x^2+9} = \frac{1}{2} \ln|x^2+9| + \text{Arctg}\,\frac{x}{3}$

$A = \left[\frac{x^2}{18} + \frac{x}{3} - \frac{1}{2}\log(x^2+9) - \text{Arctg}\left(\frac{x}{3}\right)\right]_{-3}^{0} = 0'06117\ldots$

2/ Area bounded of $e^{-x}\cos x$ $\quad x = 0$ and $x = \pi$ $\quad u\,dv = uv - \int v\,du$

$e^{-x}\cos x = 0$; $\cos x = 0$; $x = \pi/2$

Alpes



$A = \int_0^{\pi/2} e^{-x}\cos x\,dx - \int_{\pi/2}^{\pi} e^{-x}\cos x\,dx$

$u = e^{-x}$ $\quad du = -e^{-x}$

$dv = \cos x$ $\quad v = \operatorname{sen} x$

$e^{-x} \cdot \operatorname{sen} x - \int \operatorname{sen} x \cdot -e^{-x}$ $\quad u =$ $\quad d$ $\quad dv =$ $\quad v =$

$-e^{-x}\cos x - (-e^{-x}\sin(x) + \int e^{-x}$

$\int u\, dv = uv - \int v\, du$ {Alpes}

un dia vi un a vaca sin cola vestida de univorme

1/ $\int \underbrace{x^2}_{dv} \underbrace{Ln x}_{u} \, dx$; $\int u dv = uv - \int v du$

$u = Ln x \quad du = \frac{1}{x}$

$dv = x^2 \quad v = \frac{x^3}{3}$

$\Rightarrow Ln x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x} \Rightarrow \frac{Ln x \cdot x^3}{3} - \int \frac{x^3}{3x} \Rightarrow$

$\Rightarrow \frac{Ln x \cdot x^3}{3} - \frac{\frac{x^4}{4}}{\frac{3x^2}{2}} \Rightarrow \frac{Ln x \cdot x^3}{3} - \frac{2x^4}{12x^2} \Rightarrow \frac{Ln x \cdot x^3}{3} - \frac{x^4}{6x^2} \Rightarrow \frac{Ln x \cdot x^3 \cdot x^2}{18}$

2/ $\int x e^x dx$ $\quad \int u dv = uv - \int v du$ Alpes

$u = x \quad du = 1$

$dv = e^x \quad v = e^x \Rightarrow xe^x - \int e^x \cdot 1 = xe^x - e^x$

3/ $\int x^3 \, Ln 2x \, dx$ $\quad \int u\, dv = uv - \int v du$ Alpes

$u = Ln 2x \quad du = \frac{2}{2x}$

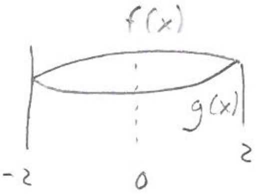
$dv = x^3 \quad v = \frac{x^4}{4}$

$\Rightarrow \frac{Ln 2x \cdot x^4}{4} - \int \frac{x^4}{4} \cdot \frac{2}{2x} \Rightarrow \text{"} - \int \frac{2x^4}{8x} \Rightarrow$

$\Rightarrow \frac{Ln 2x \cdot x^4}{4} - \frac{\frac{2x^5}{5}}{\frac{8x^2}{2}} = \text{"} - \frac{4x^5}{40x^2} = \text{"} - \frac{x^5}{10x^2}$

1/ $f(x) = \frac{16}{4+x^2}$ $\quad g(x) = |x| \Rightarrow$ Always $x \geq 0$ $\quad \frac{16}{4+x^2} = x$; $16 = 4x + x^3$; $x^3 + 4x - 16$; $x = 2$

$\frac{16}{4+x^2} = -x$; $x = -2$



$2 \int_0^2 \left(\frac{16}{4+x^2} - x \right) dx$

$\int \frac{16}{4+x^2} \Rightarrow \int \frac{16/4}{4/4 + \frac{x^2}{4}} = \frac{16}{4 \cdot 1/2} \int \frac{1/2}{1 + \left(\frac{x}{2}\right)^2} = 8 \, Arctg\left(\frac{x}{2}\right)$

$A = 2\left[8 \, arctg \frac{x}{2} - \frac{x^2}{2} \right]_0^2 = 8'57$

Simple trapezoid rule

$$\int_a^b f(x)\,dx \simeq I^* = \frac{(b-a)}{2}\cdot[f(a)+f(b)]$$

Error $\Rightarrow$ $[a,b]$; M_2 = upper band of $|f''(x)|$

$$|I - I^*| \leq \frac{M_2}{12}(b-a)^3$$

1) $I = \int_0^{\pi/2} \sin x\,dx$

$$I^* = \frac{(\pi/2 - 0)}{2}\cdot[f(\pi/2)+f(0)] = \pi/4$$

$$|I - I^*| \leq \frac{M_2\,(b-a)^3}{12} \simeq \frac{\pi/2\cdot(\pi/2-0)^3}{12} \simeq 0'323$$

$F' = \cos x$; $f'' = |-\sin x| \Rightarrow \pi/2 \uparrow$

Composite trapezoid rule (with subintervals)

$$h = \frac{b-a}{n} \qquad I^* = \frac{h}{2}\left(f(x_0) + 2f(x_1) + \dots + f(x_n)\right]$$

$$|I - I^*| \leq \frac{M_2\,(b-a)^3}{12n^2}$$

1) $I = \int_0^{\pi/2} \sin x\ dx \qquad n=3$

$$h = \frac{b-a}{n} = \frac{\pi/2 - 0}{3} = \pi/6 \qquad I = \frac{\pi}{12}\cdot\left[f(0) + 2f(0+\pi/6) + 2f(\pi/3) + f(\pi/2)\right) =$$
$0'9772$

$$\text{Error} \leq \frac{M_2\,(b-a)^3}{12n^2} = 0'018$$

$2)\ \pi = \int_0^1 \frac{4}{1+x^2}\,dx$ π up to an error of 10^{-2}

$$\frac{M_2\,(b-a)^3}{12\,n^3}$$

$$f'(x) = \frac{-8x}{(1+x^2)^2} = \frac{-8x}{1+x^4+2x^2}$$

$$f''(x) = \frac{-8(1+x^4+2x^2)+8x(4x^3+4x)}{(1+x^4+2x^2)^2} =$$

$$= -8-8x^4-16x^2+32x^4+32x^2$$

ple Simpson's rule. 3points

$$\left(a, f(a)\right),\ \left(\frac{b+a}{2}, f\left(\frac{b+a}{2}\right)\right),\ \left(b, f(b)\right)$$

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$$

$$|I-I^*| \leq \frac{M_4}{90}\left(\frac{b-a}{2}\right)^5$$ M_4 = upper bound of the $|f^{4}(x)|$

$1/\int_0^{\pi/2} \sin x\ dx \approx$

$$\left(\frac{b-a}{6}\right)\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$$

$$\approx \frac{\pi/2}{6}\cdot\left(f(0)+4f(\pi/4)+f(\pi/2)\right) \approx 1'00227\ldots$$

$f' = \cos x$
$f'' = -\operatorname{sen} x$
$f''' = -\cos x$
$f^{IV} = \operatorname{sen} x$

$$|I-I^*| = \frac{M_4}{90}\cdot\left(\frac{b-a}{2}\right)^5 = \frac{\operatorname{sen}\pi/2}{90}\cdot\left(\pi/4\right)^5 = 0'0033$$

Improper integrals

1- Unbounded

$$\int_a^\infty f(x)\ ;\ \int_{-\infty}^b f(x)\ ;\ \int_{-\infty}^\infty f(x)$$

$$a)\ \int_0^\infty e^{-x} = \left[-e^{-x}\right]_0^\infty = -e^{-\infty}+e^{-x} = 1$$

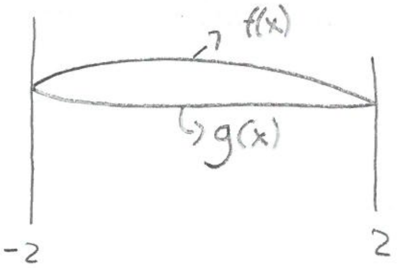
$$\lim_{t\to\infty}\int_a^t f(x)$$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

$f(x) = \frac{40}{36+x^2} \qquad g(x) = x^2 - 3$

$f(x) = g(x) \;;\; \frac{40}{36+x^2} = x^2 - 3 \;;\; 40 = 36x^2 + x^4 - 108 - 3x^2$

$x = \pm 2$



$$\int_{-2}^{2} f(x) - g(x) = \int_{-2}^{2} \frac{40}{36+x^2} - x^2 + 3$$

$$\left[\frac{20}{3} \operatorname{Arctg} \frac{x}{6} - \frac{x^3}{3} + 3x \right]_{-2}^{2}$$

$$\left[\frac{20}{3} \operatorname{Arctg} \frac{2}{6} - \frac{2^3}{3} + 3 \cdot 2 \right] - \left[\frac{20}{3} \operatorname{Arctg} \frac{-2}{6} - \frac{-2^3}{3} + 3 \cdot -2 \right) = 10'457\, u^2$$

$$\int \frac{40}{36+x^2} ; \int \frac{40/36}{36/36 + \left(\frac{x}{6}\right)^2} = \frac{10}{9} \int \frac{1}{1+\left(\frac{x}{6}\right)^2} = \frac{10}{9} \cdot \frac{6}{1} \cdot \int \frac{1/6}{1+\left(\frac{x}{6}\right)^2} d = \frac{20}{3} \operatorname{Arctg} \frac{x}{6}$$

1) $\int_{-\infty}^{0} e^x \sin 2x \, dx \qquad udv = uv - \int v du$

$+\frac{1}{2} \int \cos 2x \, e^x$

$$\lim_{t \to -\infty} \int_{-\infty}^{0} e^x \sin 2x = -\frac{e^x}{2} \cdot \cos 2x - \int -\frac{1}{2} \cdot \cos 2x \, e^x = -\frac{e^x}{2} \cdot \cos 2x + \frac{1}{2} \left(\frac{e^x}{2} \operatorname{sen} 2x - \int \frac{1}{2} \operatorname{sen} 2x \, e^x \right.$$

$u = e^x \qquad du = e^x$

$dv = \sin 2x \qquad v = -\frac{1}{2} \cos 2x$

$u = e^x \qquad du = e^x$

$dv = \cos 2x \qquad v = \frac{1}{2} \operatorname{sen} 2x$

$$-\frac{e^x}{2} \cdot \cos 2x + \frac{1}{2} \cdot \left(\frac{e^x}{2} \sin 2x - \frac{1}{2} \int \operatorname{sen} 2x \, e^x \right)$$

$u = e^x \qquad du = e^x$

$dv = \operatorname{sen} 2x \qquad v = -\frac{1}{2} \cos 2x$

1) $\int_1^2 \cos x^2$ Trapezoidal rule up to an error of 10^{-4}

$$\text{Error} \leq \frac{M_2\,(b-a)^3}{12\,n^2} < 10^{-4}$$

$f'(x) = -2x\sin x^2$

$f''(x) = -2\cdot 2x\cos x^4 = -4x\cos x^2 - 2\sin x^2$

$|f''(x)| = 4x^2|\cos x^2| + 2|\sin x^2| = 4\cdot 2^2 + 2 = 18$

$$\frac{18\,(2-1)^3}{12\,n^2} < 10^{-4}\ ;\quad 122'47 < n = 123$$

2) $I = \int_0^1 \sin x^2$

a) Approx I. by simple trapezoid rule and an upper band of the error

$$I^* = \frac{M_2\,(b-a)^3}{12} = \frac{6\,(1-0)}{12} = 1/2$$

$f' = 2x\cos x^2$

$f'' = 2\cos x^2 + 2x\cdot(-2x\,\text{sen}\,x^2)$

$= 2\cdot|\cos x^2| + 4x^2|\text{sen}\,x^2|$

$2 + 4 = 6$

Impropers

$\int_0^1 \frac{1}{\sqrt{x}}$ continuous on $(0,1]$

$$\lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}} = \left| \frac{x^{1/2}}{1/2} \right|_t^1 = 2\sqrt{1} - 2\sqrt{t} = 2\sqrt{1}$$

40) $I = \int_0^1 \sin(e^{-x})$ error $= 0'001$ composite trapezoide

$$\text{Error} = \frac{m_2 (b-a)^3}{12n^2} \leq 0'001$$

$$f'(x) = -e^{-x} \cos e^{-x}$$

$$f''(x) = e^{-x} \cdot \cos e^{-x} + -e^{-x} \cdot -e^{-x} \cdot (-\sin e^{-x})$$

$$f''(x) = e^{-x} \cos e^{-x} + e^{-2x} \cdot (-\sin e^{-x})$$

$$|f''(x)| = \cancel{1} \cdot 1 + 1 \cdot 1 = 2$$

$$\frac{2(1-0)^3}{12n^2} \leq 0'001$$

$$12'9 \leq n = 13$$

b) Composite simpsone rule   5 points $[0,1]$ $|f^4(x)| \leq 8$

$2n+1 = 5$; $n = 5-1/2 = 2$ $h = \frac{b-a}{n} = 1/2$

$$I^* = \frac{h}{6} [f(a) + 4f(x_{1n}) + 2f(x_{1n}) + \dots + f(b)]$$

$$I^* = \frac{1/2}{6} [f(0) + 4f(1/2) + f(1)] = 0'29$$

412) $I = \int_0^{\pi/2} \log(1 + \sin x)$

a) Simple trapezoide

$$I = \frac{(b-a)}{2} [f(a) + f(b)] = \frac{\pi/2 - 0}{2} [f(0) + f(\pi/2)] = 0'5443$$

Simple simpsone

$$I = \frac{(b-a)}{6} [f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)] =$$

$$= \frac{\pi/2 - 0}{6} \cdot [f(0) + 4f(\pi/4) + f(\pi/2)] = 0'7414$$

b) Error $= 0'01$ trapezoid

$$E = \frac{m_2 (b-a)^3}{12 n^2} = \frac{1 \cdot \pi/2}{12n^2} \leq 0'01$$

$$5'68 < n = 6$$

$$f'(x) = \frac{\cos x}{(1 + \sin x)}$$

$$f''(x) = \frac{-\sin x (1 + \sin x) - \cos x \cdot \cos x}{(1 + \sin x)^2} = \frac{-\sin x \overbrace{- \sin^2 x - \cos^2 x}^{-1}}{(1 + \sin x)^2}$$

$$= \frac{-\sin x - 1}{(1 + \sin x)^2} = \frac{-1}{1 + \sin x}$$

$$M_2 = 1$$

Simple Trapezoid rule

$$I^* = \frac{(b-a)}{2}\left[f(a)+f(b)\right]$$

$$\text{Error} \leq \frac{m_2}{12}(b-a)^3$$

Composite Trapezoid method

$$I^* = \frac{h}{2}\cdot\left[f(x_0)+2f(x_{1h})+\dots+f(x_n)\right]$$

$$\text{Error} \leq \frac{m_2}{12}\frac{(b-a)^3}{n^2} \qquad h = \frac{b-a}{n}$$

Simple Simpsone method

$$I^* = \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

$$\text{Error} \leq \frac{m_4}{90}\left(\frac{b-a}{2}\right)^5$$

Composite Simpsone method

$$I^* = \frac{h}{6}\left[f(x_0)+4f(x_{1h})+2f(x_{2h})+4f(x)+f(x_n)\right]$$

$$\text{Error} \leq \frac{m_4(b-a)^5}{90n^4 2^5}$$